The Solution Of Nonhomogen Abstract Cauchy Problem by Semigroup Theory of Linear Operator

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Abstract

In this article we will investigate how to solve nonhomogen degenerate Cauchy problem via theory of semigroup of linear operator. The problem is formulated in Hilbert space which can be written as direct sum of subset Ker M and Ran M*. By certain assumptions the problem can be reduced to nondegenerate Cauchy problem. And then by composition between invers of operator M and the nondegenerate problem we can transform it to canonic problem, which is easier to solve than the original problem. By taking assumption that the operator A is infinitesimal generator of semigroup, the canonic problem has a unique solution. This allow to define special operator which map the solution of canonic problem to original problem. ©2016 JNSMR UIN Walisongo. All rights reserved.

Key words: Degenerate; Nondegenerate; Cauchy Problem; Infinitesimal.

1. Introduction

Let us consider the homogen abstract Cauchy problem:

\[ \frac{d}{dt} Mz(t) = Az(t), \quad z(0) = z_0 \quad (1) \]

For finite dimensional, the problem (1) homogen abstract degenerate Cauchy problem is discussed completely in the book [1], where we can transform matrices M and A to a common normal form. We also can find many examples, applications to control theory, and references to the earlier literature in his book.

In the infinite dimensional case, it is mentioned in [2] that they treat the singular and degenerate Cauchy problem. In [3-8], the investigates degenerate Cauchy problem in Hilbert space. In his articles, the problem is treated also under the assumption that the Hilbert space of the system can be written as direct sum of the kernel of M (Ker M) and the range of adjoint M (Ran M*).

By certain asumptions [9,10] to discuss the Degenerate Cauchy problem in Banach...
space where the space can be written in direct sum of two subspaces. The Cauchy problem in Hilbert space which can be written as direct sum of the kernel of \( M \) (Ker \( M \)) and the range of adjoint \( M \) (Ran \( M^* \))[11,12]. The possibility of factorization and the relation of the factorized problem with the original degenerate system without assuming parabolicity [13].

2. Nonhomogen Abstract Cauchy Problems

Let \( H \) and \( W \) be Hilbert space [14] over \( K \) complex number, and 

\[ M : D(M) \subset H \rightarrow W ; A : D(A) \subset H \rightarrow W. \]

are linear operators. In this section we are going to investigate how to solve nonhomogen abstract degenerate Cauchy problem,

\[ \frac{d}{dt} Mz(t) = Az(t) + f(t), \quad z(0) = z_0 \quad (2) \]

where \( M \) is not invertible. Problem (2) is called degenerate when \( M \) is not invertible. The solution of (2) is defined in the following.

**Definition 1:** Function \( z : [0, \infty) \rightarrow H \) is a strict solution of problem (2) if \( z(t) \in D(A) \cap D(M) \) for all \( t \geq 0 \), \( Mz(t) \) is continuously differentiable, and (2) hold.

To solve the problem we use several assumptions:
1. The operator \( A, M \) are closed linear operator which are densely defined in some Hilbert space,
2. \( PD_A \subset D_A \) and operator \( (QAP)PD_A \) has bounded inverse,
3. Operator \( A \) has bounded inverse,
4. Set of \( D_A \) is contained in domain of operator \( M \),
5. Operator \( M \) has a closed domain, and
6. Operator \( A_1 \) is a generator of Co-Semigrup \( di H_0 \),

So the problem (2) can be reduced to nondegenerate abstract Cauchy [15,16] problem:

\[ \frac{d}{dt} Mx(t) = A_n x(t) + (Q^T - Q^T AP(QAP)^{-1} Q)(t) = A_n x(t) + Y_A f(t) \quad (3) \]

Operator \( M \) has close domain, so the problem (3) can be transformed to problem:

\[ \frac{d}{dt} x(t) = A_1 x(t) + (M_1)^{-1} Y_A f(t), \]

where \( A_1 = (M_1)^{-1} A_0. \quad (4) \]

Since the operator \( A^1 \) is a bounded operator, so we can define \( g(t) = P^T A^{-1} f(t) \) and then problem (4) can be written:

\[ \frac{d}{dt} x(t) = A_1 x(t) + g(t). \quad (5) \]

If \( g(t) \in D(A_1) \) and continuously differentiable then the solution of equation (5) is

\[ x(t) = e^{A_1 t} P^T z_0 + \int_0^t e^{A_1(t-s)} A_1 g(s) ds \]

\[ = e^{A_1 t} P^T z_0 + A_1 \int_0^t e^{A_1(t-s)} g(s) ds \]

**Lemma 2:** If function \( z(t) \) be solution of problem (2), then function \( z(t) \) can be given by

\[ z(t) = Z_A P^T z(t) - (QAP)^{-1} Qf(t), \quad \text{for all } t \geq 0. \]

**Proof:** Now we will prove the lemma by reductio ad absurdum. We know that \( z(t) \) is solution of (1) and let \( Z_A P^T z(t) - (QAP)^{-1} Qf(t) = y(t). \)

So we will have:

\[ \frac{d}{dt} My(t) = \frac{d}{dt} M \left[ Z_A P^T z(t) - (QAP)^{-1} Qf(t) \right] \]

\[ = \frac{d}{dt} Mz(t) = Az(t) + f(t). \]

Moreover, lets \( z(t) \neq y(t) \) so we have:

\[ \frac{d}{dt} Mz(t) \neq \frac{d}{dt} My(t) \neq Az(t) + f(t). \]

**Theorem 3:** Under assumptions 1, 2, 3, 4, 5 and function \( f(t) \in \text{Ran} \ M \) continuously differentiable, for each \( z_0(t) \in D_A \) the solution of problem (2) is given by
\[ z(t) = Z_A x(t) - (QAP)^{-1} Qf(t) \] where
\[ x(t) = e^{At} P^T z_0 + A_1 \int_0^t e^{A(t-s)} g(s) ds, \]
and
\[ g(s) = P^T A^{-1} f(s). \]

In order to solve nondegenerate Cauchy problem by factorization method, the operator where is located in the right handside must be generator of semigroup linear operator, so the solution of the problem is unique.

**Definisi 3:** Let \( \mathcal{H} \) be Hilbert space and \( S(t): \mathcal{H} \to \mathcal{H} \), be linear operator for \( t \in \mathbb{R}_+ \). Semigroup \( \{ S(t) \} \) is set of operators \( S(t): \mathcal{H} \to \mathcal{H} \), for all \( t \in \mathbb{R}_+ \) where

i. \( S(t+s) = S(t)S(s) \), for all \( t, s \in \mathbb{R}_+ \).

ii. \( S(0) = I \).

Moreover \( \{ S(t) \} \) is called strongly continuous or Co-semigrup, if \( \lim_{t \to 0} S(t)x = x \), \( \forall x \in \mathcal{H} \).

On the following lemma will be discuss about differentiability of function \( t \to S(t)x \), \( x \in \mathcal{H} \).

**Lemma 4:** Let \( \{ S(t) \} \) be Co-semigrup on Hilbert Space \( H \) and \( x, y \in \mathcal{H} \). Then the following two statements are equivalent:

i. \( \lim_{t \to 0} \frac{1}{t}(S(t)x-x) = y \).

ii. The function \( S(t) \) is differentiable for \( t > 0 \) and \( \frac{d}{dt} S(t)x = S(t)y \), for all \( t > 0 \).

Next, if the operator \( A: \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H} \), where

\[ \mathcal{D}(A) = \left\{ x \in \mathcal{H} \left| \lim_{h \to 0} h \left( S(h)x - x \right) \right| \right\}, \]

then we will define infinitesimal genartor of semigroup.

**Definition 5:** The Infinitesimal generator \( A: \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H} \) of semigrup \( \{ S(t) \} \) is defined by:

\[ Ax = \lim_{t \to 0} \frac{S(t)x-x}{t} \]

where \( x \in \mathcal{D}(A) \) if only if the limit above exists.

Every linear operator \( A \) in Hilbert space does not always be generator of Co-semigrup. By the Hille-Yosida theorem we know the characteristic the linear operator is a generator of Co-semigrup[18].

**Teorema 6:** (Teorema Hille-Yosida) [17] Let \( A \) be linear operator , \( A: \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H} \), \( J \geq 1 \) and \( \omega \in \mathbb{R} \) be constans. Then the following two statements are equivalent:

a. The operator \( A \) is infinitesimal generator of Co-semigrup.

b. The operator \( A \) is closed, densely defined operator and \( \| (A - \lambda I)^{-1} \| \leq \frac{J}{\text{Re}(\lambda) - \omega} \).

For all \( \lambda \in \mathcal{S} \) with 
\[ S = \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda > \omega \} \subset \rho(A). \]

**Teorema 7:** Let \( \{ S(t) \} \) be a Co-semigrup on Hilbert space \( H \) with infinitesimal generator \( A \), then \( x \in \mathcal{D}(A) \) implies

a. \( S(t)x \in \mathcal{D}(A) \), \( \forall t \geq 0 \).

b. \( S(t)x \) is continuously differentiable on \( t \geq 0 \), with

\[ \frac{d}{dt} S(t)x = AS(t)x = S(t)Ax, \quad \forall t \geq 0. \]

\( M \) is bounded, so \( M_r \) is bounded and densely defined on \( P^T \mathcal{H} \). Since \( M_r \) is invertible, the operator \( (M_r)^{-1} \) exist and bounded. Moreover we defined the operator

\[ A_t = (M_r)^{-1} A_0 \tag{6} \]

on the natural domain:

\[ \mathcal{D}(A_t) = \{ x \in P^T \mathcal{D}_A \mid A_0 x \in \text{Ran} M \} = A_0^{-1} \text{Ran} M. \]

According (6), the problem nondegenerate (5) can be tranformed to canonic form:

\[ \frac{d}{dt} x(t) = (M_r)^{-1} A_0 x(t), \quad x(0) = P^T z_0. \tag{7} \]

Operator \( A_t = (M_r)^{-1} A_0 \) on natural domain is closed, because it is the product of a bounded
operator \((M_r)^{-1}\) and a closed operator \(A_0\).
Operator \(A_0\) is densely defined in Hilbert space \(\mathcal{H}_0 = (P^T D)\).

According Theorem 7, for every \(x \in \mathcal{D}(A)\) the function \(u(t) = S(t)x\) is a solution of Cauchy abstract problem
\[
d\frac{d}{dt} u = Au, \ u(0) = x.
\]
Moreover we must have the assmuption to make the canonik problem has a unique solution.

**Asumtion 8:** \(A_t\) generates a strongly continuous semigroup in \(H_0\).

By assumption 8 dan Theorem 7, the solution problem (7) is \(x(t) = S(t)x_0\) where \(x_0 = P^T z_0\).

**Example 9:** In this example, we have Hilbert space \((L^2(R))^2\) and operator
\[
M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & a \end{pmatrix}.
\]

According (3), then we have
\[
D_A = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in (W^{1,2}(R))^2 \bigg| g(x) = -\frac{1}{a} f'(x) \right\}.
\]

Next, we define an orthogonal projection operators:
\[
P = Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{dan} \quad P^T = Q^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

By the projection operators, we have operator:
\[
M_r = MP^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_0 = Y_A A = AP^T = \begin{pmatrix} 0 & \frac{\partial^2}{\partial a^2} \\ \frac{\partial}{\partial a^2} & 0 \end{pmatrix}.
\]

In this case, operator \(A_0 = \frac{1}{a} \frac{\partial^2}{\partial a^2}\) on \(W^{2,2}(R)\) and \(M_r = 1\) on \(L^2(R)\). So the solution of this problem is
\[
z(t) = Z_A x(t), \quad \text{where} \quad x(t) = e^{At} P^T z(0), \quad Z_A = \begin{pmatrix} 1 & 0 \\ -\frac{\partial}{\partial a^2} & 0 \end{pmatrix}.
\]

**3. Conclusion**

After we discuss this topic, we can result how to solve degenerate abstract Cauchy problem by semigroup theory of linear operator. There are three stages to solve this problem. In the first stage, under certain assumptions we reduce the nonhomogenous abstract degenerate Cauchy problem (2):
\[
\frac{d}{dt} M z(t) = A z(t) + f(t), \quad z(0) = z_0
\]
to nondegenerate Cauchy problem (3):
\[
\frac{d}{dt} M_r x(t) = A_r x(t) + Y_x f(t), \quad x(t) = P^T z_0.
\]

Operator \(M\) which is not invertible, can be reduced to invertible operator \(M_r\).

The second stage the problem (3) can be tranformed into canonik form:
\[
\frac{d}{dt} x(t) = A_1 x(t) + (M_r)^{-1} Y_x f(t),
\]
where \(A_1 = (M_r)^{-1} A_0\).

In order to use semigroup theory of linear operator, we assume that \(A_1\) is an *infinitesimal* generator of strongly continuous semigroup, so the solution is
\[
x(t) = e^{At} P^T z_0 + A_1 \int_0^t e^{A(t-s)} g(s) ds,
\]

The third, finally by \(Z_A\) we can find a solution of the original problem is
\[
z(t) = Z_A x(t) - (QAP)^{-1} Q f(t)
\]
and \(g(s) = P^T A^{-1} f(s)\).
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References