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Fixed point results in α, β partial b-metric spaces using C-contraction type mapping and its generalization

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Abstract

Banach contraction mapping has main role in nonlinear analysis courses and has been modified to get new kind of generalizations in some abstract spaces to produce many fixed point theory. Fixed point theory has been proved in partial metric spaces and b-metric spaces as generalizations of metric spaces to obtain new theorems. In addition, using modified of contraction mapping we get some fixed point that have been used to solve differential equations or integral equations, and have many applications. Therefore, this area is actively studied by many researchers. The goal of this article is present and prove some fixed point theorems for extension of contraction mapping in α, β partial b-metric spaces. In this research, we learn about notions of b-metric spaces and partial metric that are combined to generated partial b-metric spaces from many literatures. Afterwards, generalizations are made to get α, β partial b-metric spaces. Using the properties of convergence, Cauchy sequences, and notions of completeness in α, β partial b-metric spaces, we prove some fixed point theorem. Fixed point theory that we generated used C-contraction mapping and its generalizations with some conditions. Existence and uniqueness of fixed point raised for some restrictions of α, β conditions. Some corollaries of main results are also proved. Our main theorems extend and increase some existence in the previous results.

Keywords: b-metric; partial metric; fixed point

1. Introduction

Partial metric spaces are notions of distance in which two similar points is not always equal to zero. This metric was studied by Matthews [1] in 1992 and come as a generalization of metric spaces. The other type of metric spaces is b-metric spaces introduced by Bakhtin [2] and extensively used by Czerwik

[3] in prove some fixed point theorem. Afterwards, Suzana Aleksic, et. al. [4] establish some new theorems for multivalued mapping in *b*-metric spaces. Also, Jain and Kaur [5] devoted some new theorems related to b-metric-like spaces and Nazam [6] construct common fixed point theorems in Partial b-metric spaces. Other researchers [7], [8], [9], [10] [11] have studied fixed point theorems involving different model of contraction mapping in b-metric space. Fixed point results in b-metric space is very active topic studied by researchers because it has many application in other fields such as in fractional differential equations, boundary value problems, or integral equations [6], [12], [13], [14] [15].

Shukla [16], in 2014, proposed partial b-metric spaces that generalize both of b-metric space and partial metric space. Shukla also prove Banach and Kannan type contraction principle. Research about partial b-metric spaces has been attracted many researchers to modified or make generalized to find new kind of fixed point theory [17], [18], [19], [20], [21]. Recently, Pravin and Virath [22] devised of α, β partial *b*-metric space (X, p) and prove Banach contraction principal in (X, p). In this paper, we shall observe Chatterjea [23] type fixed point and some other generalization in α, β partial b-metric space.

2. Methods

The method of this research is literature study. Research begins with study about theory in b-metric spaces, partial metric spaces and related fixed point. Afterwards, we learn fixed point of partial b-metric spaces as generalization of *b*-metric spaces. From article Pravin and Virath [22] , concept α,β partial *b*-metric space is defined and prove fixed point theorems. Using concept α,β partial *b*-metric space, in this article we formulate and prove main fixed point results to extends previous results.

In this part, we write some definitions, theorems and symbols that underlies main results.

Definition 2.1 [16]

Given a non-empty X set and function $b: X\times X \to \mathbf{R}$. Function b is called partial b-metric on X, if there exist a real number $\alpha \geq 1$ for all $x, y, z \in X$ hold

(i) $b(x,x)=b(y,y)=b(x,y)$ iff $x=y$

(ii) $b(x,x)\leq b(y,x)$

(iii) $b(x,y)=b(y,x)$

(iv) $b(x,y)\leq \alpha\left[b(x,z)+b(z,y)\right]-b(z,z)$

The pair (X,b) is called partial *b*-metric spaces with coefficient α.

Definition 2.2 [22]

Given a non-empty set X and function $b_p: X\times X \to \mathbf{R}$. Function b_p is α,β partial b-metric on X if there exists two real numbers $\alpha,\beta \geq 1$ for all $x, y, z \in X$ satisfied

(i) $b_p(x,x)=b_p(x,y)=b_p(y,y)$ if and only if $x=y$

(ii) $b_p(x,x)\leq b_p(x,y)$

(iii) $b_p(x,y)=b_p(y,x)$

(iv) $b_p(x,y)\leq \alpha b_p(x,z)+\beta b_p(z,y)-b_p(z,z)$

Furthermore, (X,b_p) is called an α,β partial b-metric space. Clearly, for $\alpha=\beta$ in (X,b_p),then we get (X,b_p) as partial *b*-metric space.

Example 2.1 [22]

Let $A=(1,3)$ and let $b_p: A\times A\to \mathbf{R}$ be a function with

$$b_p(x,y)=e^{|x-y|}+\tfrac{1}{2} \tag{1}$$

for every $x,y,z\in A$. Then (A,b_p) is an α,β partial b-metric space.

Every α,β partial b-metric on X have a open balls $N_{b_p}(x,\varepsilon)=\{y\in X: b_p(x,y)<\varepsilon+b_p(x,x)\}$ and $\overline{N_{b_p}(x,\varepsilon)}=\{y\in X: b_p(x,y)\leq\varepsilon+b_p(x,x)\}$ with $x\in X$ and $\varepsilon>0$.

Definition 2.3 [22]

Let (X,b_p) be an α,β partial b-metric space, and sequence $\{x_n\}\subseteq X$ and $x\in X$, we get

(i) $\{x_n\}$ converges to x, if $\lim_{n\to\infty} b_p(x_n,x)=b_p(x,x)$.

(ii) $\{x_n\}$ is a Cauchy in (X,b_p), provide $\lim_{n,m\to\infty} b_p(x_n,x_m)$, exists and finite.

Definition 2.4 [22]

The α,β partial b-metric space (X,b_p) is said complete if any Cauchy sequence $\{x_n\}$ in X satisfied

$$\lim_{n,m\to\infty} b_p(x_n,x_m)=\lim_{n\to\infty} b_p(x_n,x)=b_p(x,x)$$

for some $x\in X$.

Definition 2.5 [23]

A mapping $F:X\to X$ is said C-contraction in (X,d) is a metric space if there is exist $\lambda\in\left(0,\frac{1}{2}\right)$ such that for every $x,y\in X$ satisfied

$$d(Fx,Fy)\le\lambda\big(d(x,Fy)+d(y,Fx)\big) \tag{2}$$

Theorem 2.1 [22]

Given (X,p) be a complete α,β partial b-metric space and $F:X\to X$ a mapping such that for all $x,y\in X$ satisfied the following condition

$$b_p(Fx,Fy)\le\lambda b_p(x,y)$$

with $\lambda\in[0,1)$. Then F has unique fixed point.

3. Result and Discussion

Now, we are ready to prove fixed point results in α,β partial b-metric space. The first result related to mapping that satisfied condition in Definition 2.5.

Theorem 3.1

Given (X,b_p) be α,β partial b-metric space with complete properties and $F:X\to X$ be a mapping such that

$$b_p(Fx,Fy)\le\lambda\left[b_p(x,Fy)+b_p(y,Fx)\right] \tag{3}$$

for all x,y in X with $\lambda\in\left[0,\frac{1}{\alpha+\beta}\right)$. Then F has a unique fixed point.

Proof.

Given $x_0\in X$ and we defined sequence $\{x_n\}$ in X that satisfied

$$x_n=Fx_{n-1}=F^n x_0$$

For any $n\in\mathbf{N}$, we obtain

$$\begin{aligned}
b_p(x_{n-1},x_n)&=b_p(Fx_{n-2},Fx_{n-1})\\
&\le\lambda\left[b_p(x_{n-2},Fx_{n-1})+b_p(x_{n-1},Fx_{n-2})\right]\\
&=\lambda\left[b_p(x_{n-2},x_n)+b_p(x_{n-1},x_{n-1})\right]\\
&\le\lambda\big[\alpha b_p(x_{n-2},x_{n-1})+\beta b_p(x_{n-1},x_n)+\\
&\qquad -b_p(x_{n-1},x_{n-1})+b_p(x_{n-1},x_{n-1})\big]\\
&=\lambda\left[\alpha b_p(x_{n-2},x_{n-1})+\beta b_p(x_{n-1},x_n)\right]
\end{aligned}$$

Repeating the above process for all $n\in\mathbf{N}$, we obtain

$$\begin{aligned}
b_p(x_{n-1},x_n)&\le\frac{\alpha\lambda}{1-\beta\lambda}b_p(x_{n-2},x_{n-1})\\
&\le\left(\frac{\alpha\lambda}{1-\beta\lambda}\right)^{n-1}b_p(x_0,x_1)
\end{aligned}$$

Therefore, $\lim_{n\to\infty} p(x_{n-1},x_n)=0$ because $\frac{\alpha\lambda}{1-\beta\lambda}<1$.

Supposed $\gamma=\frac{\alpha\lambda}{1-\beta\lambda}$. For $m,n\in\mathbf{N}$,

$$
\begin{aligned}
b_p(x_n,x_{n+m}) &\le \alpha b_p(x_n,x_{n+1})+\beta b_p(x_{n+1},x_{n+m})+ \\
&\quad -b_p(x_{n+1},x_{n+1}) \\
&\le \alpha b_p(x_n,x_{n+1})+\beta b_p(x_{n+1},x_{n+m}) \\
&\le \alpha b_p(x_n,x_{n+1})+\beta\alpha b_p(x_{n+1},x_{n+2})+ \\
&\quad \beta^2\alpha b_p(x_{n+2},x_{n+3})+\cdots+ \\
&\quad \beta^{m-1}b_p(x_{n+m-1},x_{n+m}) \\
&\le \alpha\gamma^n b_p(x_0,x_1)+\beta\alpha\gamma^{n+1}b_p(x_0,x_1)+ \\
&\quad \beta^2\alpha\gamma^{n+2}b_p(x_0,x_1)+\cdots+ \\
&\quad \beta^{m-1}\gamma^{n+m-11}b_p(x_0,x_1) \\
&\le \gamma^n b_p(x_0,x_1)\left[\alpha+\alpha\beta\gamma+\alpha\beta^2\gamma^2+\right. \\
&\quad \left.\cdots+\beta^{m-1}\gamma^{m-1}\right] \\
&= \gamma^n b_p(x_0,x_1)\left[\frac{\alpha\left(1-(\beta\gamma)^{m-1}\right)}{1-\beta\gamma}+(\beta\gamma)^{m-1}\right]
\end{aligned}
$$

Since $\gamma=\frac{\alpha\lambda}{1-\beta\lambda}<1$, then $\left(\frac{\alpha\lambda}{1-\beta\lambda}\right)^n\to 0$ as $n\to\infty$ and $(\beta\gamma)^n\to 0$ for $n\to\infty$. Then implies that $\lim_{n,m\to\infty} b_p(x_n,x_{n+m})=0$. Therefore, the sequence $\{x_n\}$ is Cauchy sequence in X, i.e.

$$\lim_{n,m\to\infty} b_p(x_n,x_m)=\lim_{n\to\infty} b_p(x_n,x)=b_p(x,x)=0$$

Since X is complete α,β partial b-metric space then $\{x_n\}$ is convergence sequence, and say $\{x_n\}$ converge to $x\in X$, so

$$\lim_{n\to\infty} b_p(x_n,x)=b_p(x,x)=0$$

For any $n\in\mathbf{N}$, we obtain

$$
\begin{aligned}
b_p(x,Fx) &\le \alpha b_p(x,x_{n+1})+\beta b_p(x_{n+1},Fx)-b_p(x_{n+1},x_{n+} \\
&\le \alpha b_p(x,x_{n+1})+\beta b_p(Fx_n,Fx) \\
&\le \alpha b_p(x,x_{n+1})+\beta\lambda\left(b_p(x_n,Tx)+b_p(x,x_{n+1})\right) \\
&\le \alpha b_p(x,x_{n+1})+\beta\lambda b_p(x,x_{n+1})+\alpha\beta\lambda b_p(x_n,x \\
&\quad \beta^2\lambda b_p(x,Tx)-\beta\lambda b_p(x,x)
\end{aligned}
$$

Therefore,

$$b_p(x,Tx)\le\frac{\alpha}{1-\beta^2\lambda}b_p(x,x_{n+1})+\frac{\beta\lambda}{1-\beta^2\lambda}b_p(x,x_{n+1})+\frac{\alpha\beta\lambda}{1-\beta^2\lambda}b_p(x_n,x)$$

Taking limit for both sides, we obtained $b_p(x,Fx)=0$. From Definition 2.2, $b_p(x,x)=0$ and $b_p(Tx,Tx)=0$.

Therefore, $b_p(x,x)=b_p(Fx,Fx)=b_p(x,Fx)=0$. So, x is a fixed point of F.

We shall present the prove of uniqueness of fixed point F. Suppose that $x,y\in X$ are distinct fixed point of F. We obtain,

$$
\begin{aligned}
b_p(x,y) &= b_p(Fx,Fy) \\
&\le \lambda\left(b_p(x,Fy)+b_p(y,Fx)\right) \\
&= \lambda\left(b_p(x,y)+b_p(y,x)\right) \\
&\le 2\lambda b_p(x,y)
\end{aligned}
$$

Therefore $b_p(x,y)=0$ that implies that $x=y$. Hence, F just has one fixed point.

The following result extended fixed point theorem by Misrah et.al [24] in α,β b-metric space.

Theorem 3.2

Given (X,b_p) be α,β partial b-metric spaces with complete properties and $F:X\to X$ be a mapping such that

$$p(Tx,Ty)\le ap(x,Tx)+bp(y,Ty)+cp(x,y)$$

for every $x,y\in X$ with a,b,c are positive real numbers that satisfied $\beta(a+b+c)<1$. Then F has a unique fixed point.

Proof.

Firstly, we make a sequence $\{x_n\}\subseteq X$ such that for any $x_0\in X$ holds

$$x_n=Tx_{n-1}=T^n x_0$$

For all $n \in \mathbf{N}$, we have

$$\begin{aligned} b_p(x_n, x_{n-1}) &= b_p(Fx_{n-1}, Fx_{n-2}) \\ &\leq ab_p(x_{n-1}, Fx_{n-1}) + bb_p(x_{n-2}, Fx_{n-2}) + \\ &\quad cb_p(x_{n-2}, x_{n-1}) \\ &= ab_p(x_{n-1}, x_n) + bb_p(x_{n-2}, x_{n-1}) + cb_p(x_{n-2} \\ &= ab_p(x_{n-1}, x_n) + (b+c)b_p(x_{n-2}, x_{n-1}) \end{aligned}$$

Therefore, $b_p(x_n, x_{n-1}) \leq \frac{b+c}{1-a} b_p(x_{n-2}, x_{n-1})$. If we continued the process, we get

$$b_p(x_n, x_{n-1}) \leq \left(\frac{b+c}{1-a}\right)^{n-1} b_p(x_0, x_1)$$

For $n, m \in \mathbf{N}$, we get

$$\begin{aligned} b_p(x_n, x_{n+m}) &\leq \alpha b_p(x_n, x_{n+1}) + \beta b_p(x_{n+1}, x_{n+m}) + \\ &\quad -b_p(x_{n+1}, x_{n+1}) \\ &\leq \left(\frac{b+c}{1-a}\right)^n b_p(x_0, x_1)\left[\alpha^2\beta\left(\frac{b+c}{1-a}\right) + \right. \\ &\quad \left. \cdots + \beta^m\left(\frac{b+c}{1-a}\right)^{m-1}\right] \end{aligned}$$

Since $\beta(a+b+c) < 1$, then $a+b+c < 1$, so $\frac{b+c}{1-a} < 1$. Taking limit for both side with $n \to \infty$, we have $\frac{b+c}{1-a} \to 0$. This gives us $b_p(x_n, x_{n+m}) \to 0$ for $n, m \to \infty$. Hence, the sequence $\{x_n\}$ is Cauchy sequence in X i.e.

$$\lim_{n,m\to\infty} b_p(x_n, x_m) = \lim_{n\to\infty} b_p(x_n, x) = b_p(x, x) = 0$$

Since X is complete α, β partial *b*-metric space, then $\{x_n\}$ is convergent sequence.

Let the sequence $\{x_n\}$ convergent to x in *X*. In this step, x as a fixed point of T will be presented. From definition of α, β partial *b*-metric space, we get

$$\begin{aligned} b_p(x, Fx) &\leq \alpha b_p(x, x_n) + \beta b_p(x_n, Fx) - b_p(x_n, x_n) \\ &\leq \alpha b_p(x, x_n) + \beta b_p(Fx_{n-1}, Fx) \\ &\leq \alpha b_p(x, x_n) + \beta\left[ab_p(x_{n-1}, Fx_{n-1})\right. \\ &\quad \left. bb_p(x, Fx) + cb_p(x_{n-1}, x)\right] \end{aligned}$$

Therefore,

$$(1-\beta b)b_p(x, Fx) \leq \alpha b_p(x, x_n) + a\beta b_p(x_{n-1}, Fx_{n-1}) + c\beta b_p(x_{n-1}, x)$$

or

$$\begin{aligned} b_p(x, Fx) &\leq \frac{\alpha}{1-\beta b} b_p(x, x_n) + \frac{a\beta}{1-\beta b} b_p(x_{n-1}, Fx_{n-1}) + \\ &\quad \frac{c\beta}{1-\beta b} b_p(x_{n-1}, x) \\ &\leq \frac{\alpha}{1-\beta b} b_p(x, x_n) + \frac{a\beta}{1-\beta b} b_p(x_{n-1}, x_n) + \\ &\quad \frac{c\beta}{1-\beta b} b_p(x_{n-1}, x) \end{aligned}$$

Since $b \neq \frac{1}{\beta}$ and $\lim_{n\to\infty} b_p(x_n, x) = b_p(x, x) = 0$, we have $b_p(x, Fx) = 0$. We get $p(x, x) = 0$ and $b_p(Fx, Fx) = 0$. Thus, $Fx = x$ that give us $x \in X$ as a fixed point of F. Furthermore, prove of a unique fixed point is given. Suppose $u, x \in X$ are distinct fixed point of F. We have

$$\begin{aligned} b_p(u, x) &= b_p(Fu, Fx) \\ &\leq ab_p(u, Fu) + bb_p(y, Fx) + cb_p(u, x) \\ &\leq ab_p(u, u) + bb_p(x, x) + cb_p(u, x) \\ &\leq ab_p(u, x) + bb_p(u, x) + cb_p(u, x) \\ &\leq (a+b+c)b_p(u, x) \end{aligned}$$

Hence, $b_p(u, x) = 0$ or $x = y$. So, F just has one fixed point in X.

Corollary 3.3

If (X, b_p) be a complete α, β partial *b*-metric space and $F : X \to X$ be a mapping satisfying

$$b_p\left(F^n x, F^n y\right) \le a b_p\left(x, F^n x\right) + b b_p\left(y, F^n y\right) + c b_p(x,y)$$

for all $x, y \in X$ with a, b, c positive real numbers that satisfied $\beta(a+b+c)<1$ and $\frac{\alpha}{\beta} < \frac{b}{a}$ for some fixed , then F has one fixed point.

Proof.

Write $S = T^n$. From Theorem 3.2, then S has just one fixed point i.e. there is $u \in X$ such that $G(u) = u$. Hence, $F^n u = u$. For some $n \in \mathbf{N}$, we have

$$\begin{aligned} b_p(Fu,u) &= b_p\left(F\left(F^n u\right),u\right) \\ &= b_p\left(F^n(Fu),u\right) \\ &\le ab_p\left(Fu, F^n(Fu)\right) + bb_p\left(u, F^n u\right) + cb_p(Fu, \\ &= ab_p\left(Fu, F\left(F^n u\right)\right) + bb_p(u,u) + cb_p(Fu,u) \\ &= ab_p(Fu,Fu) + bb_p(u,u) + cb_p(Tu,u) \end{aligned}$$

Therefore,

$$\begin{aligned} (1-c)b_p(Fu,u) &\le ab_p(Fu,Fu) + bb_p(u,u) \\ &\le a\alpha b_p(Fu,u) + a\beta b_p(u,Fu) + \\ &\quad -b_p(u,u) + bb_p(u,u) \\ &= a(\alpha+\beta)b_p(Fu,u) - (1-b)b_p(u,u) \\ &\le a(\alpha+\beta)b_p(Fu,u) \end{aligned}$$

Hence, $b_p(Fu,u) \le \dfrac{a(\alpha+\beta)}{1-c} b_p(Fu,u)$, this implies $b_p(Fu,u)=0$ Thus, $Fu = u$ or $u \in X$ is fixed point of F.

Next, prove of the uniqueness is start with assume w is another a fixed point F.

$$\begin{aligned} b_p(u,w) &= b_p(Tu,Tw) \\ &= b_p\left(T^n u, T^n w\right) \\ &\le ab_p\left(u, T^n u\right) + bb_p\left(w, T^n w\right) + cb_p(u,w) \\ &= ab_p(u,u) + bb_p(w,w) + cb_p(u,w) \\ &\le ab_p(u,w) + bb_p(u,w) + cb_p(u,w) \\ &= (a+b+c)b_p(u,w) \end{aligned}$$

It is implied that $b_p(u,w)=0$ or $u=w$. So, F has a single fixed point.

The next theorem is a generalization of result from Pravin and Virath using Ciric [25] type contraction.

Theorem 3.4

If (X, b_p) be a complete α,β partial *b*-metric space and $F: X \to X$ be a mapping such that for every $x, y \in X$ satisfied

$$b_p(Fx,Fy) \le \lambda \max\{b_p(x,y), b_p(x,Fx), b_p(y,Fy), b_p(x,Fy), b_p(y,Fx)\}$$

for $\lambda \in \left[0, \frac{1}{\beta}\right)$, then F has a unique fixed point.

Proof.

Define a sequence in X such that

$$x_n = Tx_{n-1} = T^n x_0$$

with $x_0 \in X$.

For $n \in \mathbf{N}$, we obtain

$$\begin{aligned} b_p(x_{n+1}, x_n) &= b_p(Fx_n, Fx_{n-1}) \\ &\le \lambda \max\{b_p(x_n, x_{n-1}), b_p(x_n, Fx_n), \\ &\qquad b_p(x_{n-1}, Fx_{n-1}), b_p(x_n, Fx_{n-1}), \\ &\qquad b_p(x_{n-1}, Fx_n)\} \\ &= \lambda \max\{b_p(x_n, x_{n-1}), b_p(x_n, x_{n+1}), \\ &\qquad b_p(x_{n-1}, x_n), b_p(x_n, x_n), b_p(x_{n-1}, x_{n+1})\} \\ &\le \lambda \max\{\{b_p(x_n, x_{n-1}), b_p(x_n, x_{n+1}), \\ &\qquad b_p(x_{n-1}, x_{n+1})\} \end{aligned}$$

If

$$\max\{b_p(x_n, x_{n-1}), b_p(x_n, x_{n+1}), b_p(x_{n-1}, x_{n+1})\} = b_p(x_n, x_{n+1})$$

then $b_p(x_{n+1}, x_n) \le \lambda b_p(x_n, x_{n+1}) < b_p(x_n, x_{n+1})$. We got contradiction, so we have

$$b_p(x_{n+1}, x_n) \le \lambda \max\{b_p(x_n, x_{n-1}), b_p(x_{n-1}, x_{n+1})\}$$

Now, we divide into two cases.

Case 1:

$\max\left\{b_p\left(x_n,x_{n-1}\right),b_p\left(x_{n-1},x_{n+1}\right)\right\}=b_p\left(x_n,x_{n-1}\right)$.

We get

$$b_p\left(x_{n+1},x_n\right)\le \lambda b_p\left(x_n,x_{n-1}\right)$$

Continuing the process, we have

$$b_p\left(x_{n+1},x_n\right)\le \lambda^n b_p\left(x_1,x_0\right)$$

Following the step in Theorem 3.1, we can establish that for $m,n\in\mathbf{N}$

$$b_p\left(x_n,x_{n+m}\right)\le \lambda^n \frac{\alpha}{1-\beta\lambda} b_p\left(x_0,x_1\right)$$

Since $\lambda\in\left[0,\frac{1}{\beta}\right)$ and $\beta\ge 1$, then $\lim_{n\to\infty} b_p\left(x_n,x_{n+m}\right)=0$. Thus, $\left\{x_n\right\}$ is Cauchy sequence in X. Because X is complete α,β partial *b*-metric space, then $\left\{x_n\right\}$ converge to $x\in X$ i.e.

$$\lim_{n\to\infty} b_p\left(x_n,x\right)=b_p\left(x,x\right)$$

We shall prove that x is a fixed point of mapping F. For $n\in\mathbf{N}$, we have

$$\begin{aligned} b_p\left(Fx,x\right) &\le \alpha b_p\left(x,x_n\right)+\beta b_p\left(x_n,Fx\right)-b_p\left(x_n,x_n\right)\\ &\le \alpha b_p\left(x,x_n\right)+\beta b_p\left(Fx_{n-1},Fx\right)\\ &\le \alpha b_p\left(x,x_n\right)+\beta\lambda b_p\left(x_{n-1},x\right)\end{aligned}$$

Therefore, $b_p\left(x,Fx\right)=0$ or $x\in X$ is a fixed point of F.

Case 2:

$\max\left\{b_p\left(x_n,x_{n-1}\right),b_p\left(x_{n-1},x_{n+1}\right)\right\}=b_p\left(x_{n-1},x_{n+1}\right)$

We get

$$\begin{aligned} b_p\left(x_{n+1},x_n\right) &\le \lambda b_p\left(x_{n-1},x_{n+1}\right)\\ &\le \lambda\alpha b_p\left(x_{n-1},x_n\right)+\lambda\beta b_p\left(x_n,x_{n+1}\right)\end{aligned}$$

We have

$$b_p\left(x_{n+1},x_n\right)\le \frac{\lambda\alpha}{1-\lambda\beta} b_p\left(x_{n-1},x_n\right)$$

Iterating the process, we obtain

$$b_p\left(x_{n+1},x_n\right)\le \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^n b_p\left(x_0,x_1\right)$$

Hence, $\lim_{n\to\infty} b_p\left(x_n,x_{n+m}\right)=0$. So, the sequence $\left\{x_n\right\}$ is Cauchy in X. Therefore, there is $x\in X$ such that $\lim_{n\to\infty} b_p\left(x_n,x\right)=b_p\left(x,x\right)$. Therefore, we have

$$\begin{aligned} b_p\left(x,Tx\right) &\le \alpha b_p\left(x,x_{n+1}\right)+\beta b_p\left(x_{n+1},Tx\right)-b_p\left(x_{n+1},x_{n+1}\right)\\ &\le \alpha b_p\left(x,x_{n+1}\right)+\beta\lambda b_p\left(x,Tx_n\right)\\ &\le \left(\alpha+\beta\lambda\right)b_p\left(x,x_{n+1}\right)\end{aligned}$$

Since $\lim_{n\to\infty} b_p\left(x_n,x\right)=b_p\left(x,x\right)$, then $p\left(x,Tx\right)=0$. Thus, x is a fixed point of F. From two case, we know that F has a fixed point. Next, suppose that F has two fixed point. We have

$$\begin{aligned} b_p\left(x,y\right) &= b_p\left(Fx,Fy\right)\\ &\le \lambda \max\left\{b_p\left(x,y\right),b_p\left(x,Fx\right),b_p\left(y,Fy\right),\right.\\ &\qquad \left. b_p\left(x,Fy\right),b_p\left(y,Fx\right)\right\}\\ &= \lambda \max\left\{b_p\left(x,y\right),b_p\left(x,x\right),b_p\left(y,y\right),\right.\\ &\qquad \left. b_p\left(x,y\right),b_p\left(y,x\right)\right\}\\ &\le \lambda \max\left\{b_p\left(x,y\right),b_p\left(x,x\right),b_p\left(y,y\right)\right\}\\ &\le \lambda b_p\left(x,y\right)\\ &< b_p\left(x,y\right)\end{aligned}$$

We have contradiction. Thus, $b_p\left(x,y\right)=0$ or $x=y$.

Corollary 3.5

If $\left(X,b_p\right)$ be a complete α,β partial *b*-metric space and $F:X\to X$ be a mapping such that

$$b_p\left(Fx,Fy\right)\le \lambda \max\left\{b_p\left(x,y\right),b_p\left(x,Fy\right),b_p\left(y,Fx\right)\right\}$$

for all $x,y\in X$ with $\lambda\in\left[0,\frac{1}{\beta}\right)$, then T has single fixed point.

4. Conclusion

Through this article, we give some notions of partial b-metric spaces and related theorems about α,β partial *b*-metric spaces. Our main results are some fixed point theorems in α,β partial *b*-metric spaces using C-contraction mapping and its modification. Fixed point is formulated with some conditions and proved to get extended from previous results. Existence and uniqueness of fixed point are proved using convergence, Cauchy sequence, and completeness in α,β partial *b*-metric space. In addition, some corollaries from the main results are given and proved.

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References

[1] S. G. Matthews, "Partial Metric Topology," *Ann N Y Acad Sci*, vol. 728, no. 1, pp. 183–197, Nov. 1994, doi: 10.1111/j.1749-6632.1994.tb44144.x.

[2] I. A. Bakhtin, "The Contraction Mapping Principle in Quasi-Metric Spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[3] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–115, 1993, [Online]. Available: http://dml.cz/dmlcz/120469%0Ahttp://project.dml.cz

[4] S. Aleksić, H. Huang, Z. D. Mitrović, and S. Radenović, "Remarks on some fixed point results in b-metric spaces," *J. Fixed Point Theory Appl.*, vol. 20, no. 147, Nov. 2018, doi: 10.1007/s11784-018-0626-2.

[5] K. Jain and J. Kaur, "Some Fixed Point Results in B-Metric Spaces," *axioms*, vol. 10, no. 55, pp. 1–15, 2021, doi: 10.14445/22315373/ijmtt-v55p506.

[6] M. Nazam, Z. Hamid, H. al Sulami, and A. Hussain, "Common Fixed-Point Theorems in the Partial b-Metric Spaces and an Application to the System of Boundary Value Problems," *Journal of Function Spaces*, vol. 2021, 2021, doi: 10.1155/2021/7777754.

[7] A. Gupta and P. Gautam, "Quasi-partial b-metric spaces and some related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2015, no. 18, 2015, doi: 10.1186/s13663-015-0260-2.

[8] H. Aydi and S. Czerwik, "Fixed Point Theorems in Generalized b-Metric Spaces," *Modern Discrete Mathematics and Analysis*, vol. 131, pp. 1–9, 2018, doi: 10.1007/978-3-319-74325-7_1.

[9] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Some common fixed point results in ordered partial b-metric spaces," 2013. [Online]. Available: http://www.journalofinequalitiesandapplications.com/content/2013/1/562

[10] T. Rashid, M. M. M. Jaradat, Q. H. Khan, Z. D. Mitrović, H. Aydi, and Z. Mustafa, "A new approach in the context of ordered incomplete partial b-metric spaces," *Open Mathematics*, vol. 18, no. 1, pp. 996–1005, Jan. 2020, doi: 10.1515/math-2020-0054.

[11] P. Gautam, L. M. Sánchez Ruiz, S. Verma, and G. Gupta, "Common Fixed Point Results on Generalized Weak Compatible Mapping in Quasi-Partial b-Metric Space," *Journal of Mathematics*, vol. 2021, 2021, doi: 10.1155/2021/5526801.

[12] H. Afshari, "Solution of fractional differential equations in quasi-b-metric and b-metric-like spaces," *Advances in Difference Equations*, vol. 2019, no. 285. Springer International Publishing, Dec. 01, 2019. doi: 10.1186/s13662-019-2227-9.

[13] H. Huang and S. Xu, "Fixed point theorems of contractive mappings in cone b-metric spaces and applications," 2013. [Online]. Available: http://www.fixedpointtheoryandapplications.com/content/2013/1/112

[14] M. B. Zada, M. Sarwar, and C. Tunc, "Fixed point theorems in b-metric spaces and their applications to non-linear fractional differential and integral equations," *J. Fixed Point Theory Appl.*, vol. 20, no. 1, pp. 1–19, Mar. 2018, doi: 10.1007/s11784-018-0510-0.

[15] Z. D. Mitrović, "Fixed point results in b-metric space," *Fixed Point Theory*, vol. 20, no. 2, pp. 559–566, 2019, doi: 10.24193/fpt-ro.2019.2.36.

[16] S. Shukla, "Partial b-Metric Spaces and Fixed Point Theorems," *Mediterranean Journal of Mathematics*, vol. 11, no. 2, pp. 703–711, 2014, doi: 10.1007/s00009-013-0327-4.

[17] J. Zhou, D. Zheng, and G. Zhang, "Fixed point theorems in partial b-metric spaces," *Applied Mathematical Sciences*, vol. 12, no. 13, pp. 617–624, 2018, doi: 10.12988/ams.2018.8460.

[18] X. Fan, "Fixed point theorems for cyclic mappings in quasi-partial b-metric spaces," *Journal of Nonlinear Science and Applications*, vol. 9, no. 5, pp. 2175–2189, 2016, doi: 10.22436/jnsa.009.05.22.

[19] N. van Dung, V. Thi, and L. Hang, "Remarks on Partial b-metric Spaces and Fixed Point Theorems," *MATEMATIˇCKI VESNIK*, vol. 4, no. December, pp. 231–240, 2017.

[20] V. Parvaneh and Z. Kadelburg, "Extended partial b-metric spaces and some fixed point results," *Filomat*, vol. 32, no. 8, pp. 2837–2850, 2018, doi: 10.2298/FIL1808837P.

[21] A. M. Hashim and H. A. Bakry, "Fixed points theorems for ciric ' mappings in partial b-metric space," *Basrah Journal of Science*, vol. 37, no. 1, pp. 16–24, 2019, doi: 10.29072/basjs.20190102.

[22] S. Pravin and S. Virath, "A Generalization of a Partial b-Metric and Fixed Point Theorems," *Aust. J. Math. Anal. Appl.*, vol. 19, no. 1, pp. 1–8, 2022.

[23] S. K. Chatterjea, "Fixed Point Theorems," *C.R. ACad., Bulgare Sci*, vol. 25, pp. 727–730, 1972.

[24] P. Kumar Mishra, S. Sachdeva, and S. K. Banerjee, "Some Fixed Point Theorems in b-metric Space," *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 1, pp. 19–22, 2014, doi: 10.12691/tjant-2-1-5.

[25] L. . B. . Ćirić, "A Generalization of Banach's Contraction Principle," *Proceedings of The American Mathematical Society*, vol. 45, no. 2, pp. 267–273, 1974.