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Second kind Chebyshev collocation technique for Volterra-Fredholm fractional order integro-differential equations

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Abstract

In this work, we present the numerical solution of fractional order Volterra–Fredholm integro-differential equations using the second kind of Chebyshev collocation technique. First, we transformed the problem into a system of linear algebraic equations, which are then solved using matrix inversion to obtain the unknown constants. Furthermore, numerical examples are used to outline the method's accuracy and efficiency using tables and figures. The results show that the method performed better in terms of improving accuracy and requiring less rigorous work.

Keywords: Approximate solutions; Bernstein collocation technique; Caputo derivative; fractional order integro-differential equations; Volterra-Fredholm

1. Introduction

This work focuses on fractional calculus, which is calculus with fractional derivatives. The ideal situation is to have the first derivative, velocity, and the second derivative, acceleration, and to be able to have any derivative between the first and second derivatives. It was discovered by Leibniz in the year 1695, a few years after discovering ordinary calculus, according to Adam [1], Caputo [2], Momani and Qaralleh [3], and Samko et al. [4], but it was later forgotten because the formula for these fractional derivatives is complex, making it difficult to work with ordinary pencil and paper, but now that we have computers and machines running, complexity is no longer a problem. The best way to model anomalous phenomena, such as heat spreading in a furnace, plasma, or the flow of water beneath the ground, is with fractional calculus. It is also used to simulate virus spread, satellite disposition in space, and system memory behavior.

Since fractional calculus has piqued the interest of mathematicians and other scientists, the solutions of fractional differential and Fractional Volterra-Fredholm Integro-Differential Equations (FVFIDEs) have received extensive attention in recent years. Because many FVFIDEs cannot be solved analytically, obtaining good approximations using numerical techniques will be extremely helpful. Many authors have presented numerical methods for solving the FVFIDEs, including the following: Mittal and Nigam [5] used the Adomian decomposition method (ADM) to solve Fractional Integro-Differential Equations (FIDEs), and Osama and Sarmad [6] used Bernstein polynomials as basis functions to approximate the solution of FIDEs. Mohammed [7] and Mahdy and Mohamed [8] presented the Least Squares Method (LSM) for solving FIDEs. Dilek and Aysegul [9] and Oyedepo et al. [10] used the collocation method for solving FIDEs. Aysegul and Dilek [11] used Lagurre polynomials as a basis, and Alkan and Hatipoglu [12] presented fractional order approximations to FVFIDEs. Mohyud-Din et al. [13] used the Chebyshev wavelet method to solve nonlinear FVFIDEs with mixed boundary conditions.

Zhou and Xu [14] introduced numerical solution of FVFIDEs with mixed boundary conditions using the Chebyshev wavelet method; Dehestani et al. [15] used a combination of Lucas wavelets and Legendre-Gauss quadrature; Salman and Mustafa [16] used Lagrange polynomials; Rajagopal et al. [17] applied a new numerical method for FIDEs; Lotfi and Alipanah [18] employed the Legendre spectral element method for solving Volterra-integro differential equations. Also, Meng et al. [19], Loh et al. [20], Keshavarz et al. [21], Ordokhani and Dehestani [22], Ordokhani and Rahimi [23], Oyedepo et al. [24-25] and Bhrawya et al. [26] contain a number of numerical techniques for solving the FIDEs.

Motivated and inspired by the preceding work, we propose a second-kind Chebyshev collocation technique with improving accuracy and less rigorous work for FVFIDEs. In this work, the fractional derivative for the problem under consideration is taken for Different values of $\propto$ yielding various approximate solutions. The class of problem studied in this work is:

$$\mu_2\varphi''(x) + \mu_1\varphi'(x) + \mu_\alpha D^\alpha\varphi(x) + \mu_0\varphi(x) = f(x) + \lambda_1\int_0^x k_1(x,t)\varphi(t)dt + \lambda_2\int_0^x k_2(x,t)\varphi(t)dt \quad (1)$$

Subject to this boundary conditions

$$\varphi(a) = 0, \varphi(b) = 0, \quad a < x < b \quad (2)$$

Where $D^\alpha\varphi(x)$ indicates the $\propto th$ Caputo fractional derivative of $\varphi(x)$, $k_1(x,t)$ and $k_2(x,t)$ are the Fredholm and Volterra intergral kernel functions, μ_1, μ_2, μ_α , λ_1 and λ_2 are known constants, $f(x)$ is a known function and $\varphi(x)$ is the unknown function to be determined.

2. Basic definitions

2.1 Riemann-Liouville fractional derivative

Riemann-Liouville fractional derivative defined as [27]:

$$D^\alpha f(x) = \frac{1}{\Gamma(r-\propto)}\int_0^x (x-s)^{r-\propto-1} f^r(s)ds, \quad (3)$$

n is positive integer with the property that $r - 1 < \propto < r$. For example if $0 < \propto < 1$ the caputo fractional derivative is

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\propto)}\int_0^x (x-s)^{-\propto} f^1(s)ds \quad (4)$$

Hence, we have the following properties:

(1) $J^\alpha J^v f = j^{\alpha+v} f, \alpha, v > 0, f \in C_\mu,$

(2) $J^\alpha x^\gamma = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma,}\ \alpha > 0, \gamma > -1, x > 0$

(3) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0)\frac{x^k}{k!},$
$x > 0, r-1 < \alpha \le r$

(4) $D^\alpha J^\alpha f(x) = f(x), \quad x > 0, n-1 < \alpha \le n,$

(5) $D^\alpha C = 0, C$ is the constant

(6)

$$\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \ge [\alpha], \end{cases}$$

where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0,1.2,\dots\}$

2.2. Chebyshev Polynomials of the Second Kind

The Chebyshev Polynomials of the second kind are defined by:

$\varphi_n(x) = \frac{\sin[(n+1)cos^{-1}x]}{sin(cos^{-1}x)}; n = 0,1,2,\ldots$ with $\varphi_0(x) = 1$ and $\varphi_1(x) = 2x$.

These polynomials form an orthogonal system with weight function $w(x) = \sqrt{1-x^2}$ on interval $[-1,1]$.

The recurrence relation is given by $\varphi_{n+1}(x) = 2x\varphi_n(x) - \varphi_{n-1}(x), \varphi_0(x) = 1, \varphi_1(x) = 2x, \varphi_2(x) = 4x^2 - 1, \varphi_3(x) = 8x^3 - 4x, n = 0,1,2,\ldots$

The shifted equivalent of it that valid in ∈ [0,1] are given as:
$\varphi_0{}^*(x) = 1, \varphi_1{}^*(x) = 4x - 2, \varphi_2{}^*(x) = 16x^2 - 16x + 3, \varphi_0{}^*(x) = 64x^3 - 96x^2 + 40x - 4$

2.3 Absolute Error

In this work, we defined absolute error as:
Absolute Error=$|\Phi(x) - \varphi(x)|$; $0 \le x \le 1$, (5)
where $\Phi(x)$ is the exact solution and $\varphi(x)$ is the approximate solution.

3. Solution of Fractional Fredholm and Volterra Integro-Differential Equations

The techniques is based on approximating the unknown functions $\varphi(x)$ as

$$\varphi(x) = \sum_i^n \varphi_i{}^*(x)c_i \tag{6}$$

Where $\varphi_i{}^*(x)$ is shifted Chebyshev polynomial of the second kind and $c_i, i = 1,2,\cdots n$ are constants. Substituting Equation (4) and also applying Equation (3) gives

$$\mu_2 \sum_i^n \varphi_i{}''^*(x)c_i + \mu_1 \sum_i^n \varphi_i{}'^*(x)c_i + \mu_\alpha\left(\frac{1}{\Gamma(r-\propto)}\int_0^x (x-t)^{r-\propto-1}\sum_i^n \varphi_i{}^{r*}(t)c_i\,dt\right) + \mu_0 \sum_i^n \varphi_i{}^*(x)c_i - \lambda_1\int_0^x k_1(x,t)\sum_i^n \varphi_i{}^*(t)c_i\,dt - \lambda_2\int_0^x k_2(x,t)\sum_i^n \varphi_i{}^*(t)c_i\,dt = f(x) \tag{7}$$

Let
$\zeta(x) = \mu_\alpha\left(\frac{1}{\Gamma(r-\propto)}\int_0^x (x-t)^{r-\propto-1}\sum_i^n \varphi_i{}^{r*}(t)c_i\,dt\right)$, $\eta(x) = \lambda_1\int_0^x k_1(x,t)\sum_i^n \varphi_i{}^*(t)c_i\,dt$ and $\tau(x) = \lambda_2\int_0^x k_2(x,t)\sum_i^n \varphi_i{}^*(t)c_i\,dt$.

Substituting $\zeta(x)$, $\eta(x)$ and $\tau(x)$ in equation (5), equation (5) becomes

$$\mu_2 \sum_i^n \varphi_i{}''^*(x)c_i + \mu_1 \sum_i^n \varphi_i{}'^*(x)c_i + \zeta(x) + \mu_0 \sum_i^n \varphi_i{}^*(x)c_i - \eta(x) - \tau(x) = f(x) \tag{8}$$

Collocating Equation (6) at equally spaced point $x_i = a + \frac{(b-a)i}{n+1}$, $[i = 1(1)(n+1)]$ gives linear system algebraic of equations in $(n+1)$ unknown constants $c'_i s$. Additional two equations are obtained from Equation (2), which are represented in matrix form:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & A_{24} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & A_{m4} & \cdots & A_{mn} \\ A_{11}{}^* & A_{12}{}^* & A_{13}{}^* & A_{14}{}^* & \cdots & A_{1n}{}^* \\ A_{21}{}^* & A_{22}{}^* & A_{23}{}^* & A_{24}{}^* & \vdots & A_{2n}{}^* \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} B_{11} \\ B_{21} \\ \vdots \\ \vdots \\ B_{mn} \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

where A_{is} and $A_{is}{}^*$ are the coefficients of c_{is} given as:

$A_{11}, A_{12}, A_{13}, \cdots A_{1n} = \mu_2 \sum_i^n \varphi_i''^*(x_1) + \mu_1 \sum_i^n \varphi_i'^*(x_1) + \zeta(x_1) + \mu_0 u(x_1) - \eta(x_1) - \tau(x_1)$,

$A_{21}, A_{22}, A_{23}, \cdots A_{2n} = \mu_2 \sum_i^n \varphi_i''^*(x_2) + \mu_1 \sum_i^n \varphi_i'^*(x_2) + \zeta(x_2) + \mu_0 u(x_2) - \eta(x_2) - \tau(x_2)$,

$A_{31}, A_{32}, A_{33}, \cdots A_{3n} = \mu_2 \sum_i^n \varphi_i''^*(x_3) + \mu_1 \sum_i^n \varphi_i'^*(x_3) + \zeta(x_3) + \mu_0 u(x_3) - \eta(x_3) - \tau(x_3)$, $A_{11}{}^*, A_{12}{}^*, A_{13}{}^*, \cdots A_{1n}{}^* = \sum_i^n \varphi_i{}^*(a)c_i\,\varphi(a)$, $A_{11}{}^*, A_{12}{}^*, A_{13}{}^*, \cdots A_{1n}{}^* = \sum_i^n \varphi_i{}^*(b)c_i$

and B_{is} are values of $f(x_i)$. The system of equations is then solved using the matrix inversion to find the unknown constants.

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & A_{24} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & A_{m4} & \cdots & A_{mn} \\ A_{11}{}^{*} & A_{12}{}^{*} & A_{13}{}^{*} & A_{14}{}^{*} & \cdots & A_{1n}{}^{*} \\ A_{21}{}^{*} & A_{22}{}^{*} & A_{23}{}^{*} & A_{24}{}^{*} & \vdots & A_{2n}{}^{*} \end{pmatrix}^{-1} \begin{pmatrix} B_{11} \\ B_{21} \\ \vdots \\ \vdots \\ B_{mn} \\ 0 \\ 0 \end{pmatrix} \quad (10)$$

Solving Equation (10) to obtain the unknown constants' values, which are then substituted back into the assumed approximate solution to obtain the required approximate solution.

4. Numerical Examples

Example 4.1

Consider the following fractional Integro-differential [12]

$$\varphi''(x) + \mu_1\varphi'(x) + \frac{1}{x}D^{\alpha}\varphi(x) + \frac{1}{x^2}\varphi(x) - \int_0^x \sin(x-t)\,\varphi(t)dt - \int_0^1 \cos(x-t)\,\varphi(t)dt = 5 + 1.50451x^{0.5} - 13x - 1.80541x^{1.5} - x^2 + x^3 - 2.067x\cos x + 5.95385\sin x \quad (11)$$

Subject to the boundary conditions $\varphi(0) = 0, \varphi(1) = 0$. For $\alpha = 0.5$, the exact solution is $\varphi(x) = x^2 - x^3$.

Applying the proposed technique for different value $\propto = 0.25, 0.5, 0.75, 1$, we have the following approximate solutions.

For $\propto = 0.25, \varphi(x) = 0.001099913664 - 0.0781782012x + 1.258075702x^2 - 1.505546197x^3 + 0.3899076354x^4 - 0.09839363348x^5$

For $\propto = 0.5$, $\varphi(x) = -3.477493347 \times 10^{-7} + 8.911 \times 10^{-7} + 1.000001733x^2 - 1.000007095x^3 + 0.000005734436341x^4 - 0.000001653993763x^5$

For $\propto = 0.75, \varphi(x) = 0.02790181811 - 0.2384448638x + 1.365160170x^2 - 1.094467024x^3 - 0.05947024762x^4 + 0.01805285073x^5$

For $\propto = 1, \varphi(x) = 0.04180385714 - 0.4988617689x + 1.829826306x^2 - 1.428246445x^3 + 0.1000924785x^4 - 0.0318246893x^5$

Example 4.2

Consider the following fractional Integro-differential [12]

$$\varphi''(x) + D^{\alpha}\varphi(x) - 2\int_0^x (x-t)\varphi(t)dt - \int_0^1 (x^2 - t)\varphi(t)dt = \frac{1}{30} - 6x - \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15} \quad (12)$$

Subject to the boundary conditions $\varphi(0) = 0, \varphi(1) = 0$. For $\alpha = 1$, the exact solution is $\varphi(x) = x^4 - x^3$. Applying the proposed technique for different values $\propto = 0.25, 0.5, 0.75, 1$, we have the following approximate solutions.

For $\propto = 0.25, \varphi(x) = -4.612508 \times 10^{-7} + 0.0354158408x + 0.033656421x^2 - 1.111888372x^3 + 0.9068757514x^4 + 0.1359420178x^5$

For $\propto = 0.5, \varphi(x) = -1.494852 \times 10^{-7} + 0.0408296333x + 0.022510945x^2 - 1.077212455x^3 + 0.8578247006x^4 + 0.1560480913x^5$

For $\propto = 0.57, \varphi(x) = 3.171 \times 10^{-9} + 0.0466614758x + 0.022510945x^2 - 1.037081915x^3 + 0.8017927517x^4 + 0.1788627557x^5$

For $\propto = 1, \varphi(x) = x^4 - x^3$

5. Result and Discussion

Table 1 Comparison of the absolute errors for example 4.1

x	[12] Error N=32	Our Method N=5
0.0	-	3.477×10^{-7}
0.2	2.048×10^{-05}	1.483×10^{-07}
0.4	2.503×10^{-05}	3.823×10^{-08}
0.6	1.789×10^{-05}	1.070×10^{-07}
0.8	7.682×10^{-05}	3.515×10^{-07}
1.0	-	7.836×10^{-07}

Table 2 Comparison of the absolute errors for example 4.1

x	Exact	Approximate	Error
0.0	0.0000	0.0000	0.0000
0.2	-0.0064	-0.0064	0.0000
0.4	-0.0384	-0.0384	0.0000
0.6	-0.0684	-0.0684	0.0000
0.8	-0.1024	-0.1024	0.0000
1.0	0.0000	0.0000	0.0000

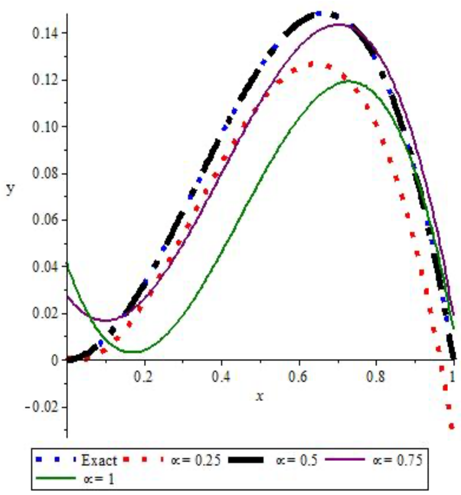

Figure 1 Showing the graphical behavior of the approximation solutions of example 4.1

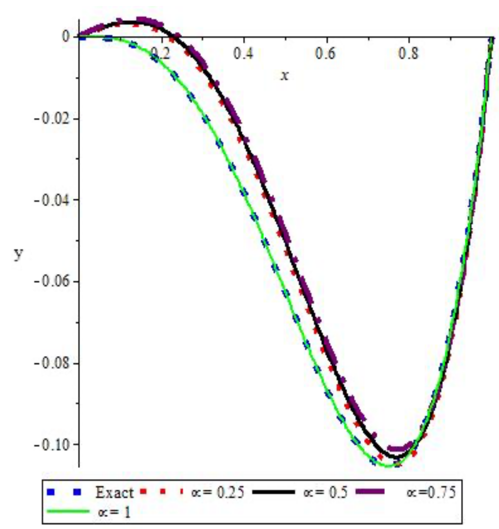

Figure 2 Showing the graphical behavior of the approximation solutions of example 4.2

Table 1 shows that our method performed more accurately because the table of errors found is smaller than [12], and it can also be seen that we got a better result at N = 5 against their result at N = 32. Figure 1 shows that at, $\propto = 0.5$, the approximate solution is in excellent agreement with the exact solution, and for $\propto = 0.25, 0.75$ and 1, the approximate solutions deviates from the exact solution as the value N of increases. Table 2 shows that our method provided an exact solution.

Figure 2 shows that at, the approximate solution is in excellent agreement with the exact solution, and for and $\propto = 0.25, 0.5$ and 0.75, the approximate solution deviates with a small change from the exact solution as the values of increase.

6. Conclusion

This work concentrated on numerical solution of FVFIDEs using second kind Chebyshev collocation technique. We confirmed that the proposed method is in excellent agreement with the exact solution using numerical calculations; Tables 1 and 2 show the effectiveness of the proposed second kind Chebyshev collocation technique over the Alkan and Hatipoglu [12] method. Based on their findings, the researchers can apply this technique to other FVFIDEs. .

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Conflicts of Interest

The authors declare no conflict of interest.

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