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Complete purely algebraic proof of the homomorphism between $SU(2)$ and $SO(3)$ without concerning their topological properties

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Abstract

The aim of this paper is to provide a complete purely algebraic proof of homomorphism between $SU(2)$ and $SO(3)$ without concerning the topology of both groups. The proof is started by introducing a map $\varphi : SU(2) \to ML(3, \mathbb{C})$ defined as $[\varphi(U)]^i{}_j \equiv \frac{1}{2} tr(\sigma_i U \sigma_j U^\dagger)$. Firstly we proof that the map φ satisfies $[\varphi(U_1 U_2)]^i{}_j = [\varphi(U_1)]^i{}_k [\varphi(U_2)]^k{}_j$, for every $U_1, U_2 \in SU(2)$. The next step is to show that the collection of $\varphi(U)$ is having orthogonal property and every $\varphi(U)$ has determinant of 1. After that, we proof that $\varphi(\mathbb{I}_2) = \mathbb{I}_3$. Finally, to make sure that φ is indeed a homomorphism, not an isomorphism, we proof that $\varphi(-U) = \varphi(U)$, $\forall U \in SU(2)$.

Keywords **:** homomorphism, rotation group, orthogonal group

1. Introduction

The study of rotation groups has been carried out in both mathematics and physics and in various topics. Discrete rotation has been studied in [14, 16]. Topologically and geometrically, rotation group has been studied in [17, 22, 8, 20, 18, 7, 19]. In representation theory, rotation groups has been studied in [2, 16, 1, 4, 15]. In physics, the application of rotation group has been discussed in [24, 1, 11, 10, 25].

Any transformation in a vector space are classified as rotation transformation if it preserve the norm of vectors in that vector space. Some authors called it as orthogonal transformations [12]. The set of rotation transformations in a vector space usually form a group classified as rotation group.

As an example, if we let the vector space is $\mathbb{R}^3$, then the rotation group in that space is $SO(3)$, which is defined by[24]

$$SO(3) \equiv \{A \in GL(3, \mathbb{R}) | AA^T = A^T A = \mathbb{I}_3\}. \quad (1)$$

As another example, if we let the vector space is $\mathbb{H}$, a space of all 2×2 complex hermitian traceless matrices, that is[24]

$$\mathbb{H} \equiv \{H \in ML(2, \mathbb{C}) | H^\dagger = H \text{ dan } tr(H) = 0\}, \quad (2)$$

then the rotation group is $SU(2)$ which is defined by[24]

$$SU(2) \equiv \{U \in GL(2, \mathbb{C}) | U^\dagger U = UU^\dagger = \mathbb{I}_2 \text{ and } det(U) = 1\}. \quad (3)$$

We also know that one of the topological properties of $SU(2)$ is simply connectedness [9]. Meanwhile, $SO(3)$ is not a simply connected topological group[23].

In various literatures discussing groups $SU(2)$ and $SO(3)$, we usually find a statetement that there is a homomorphism from $SU(2)$ to $SO(3)$. The homomorphism of group $SU(2)$ to $SO(3)$ play an important role in quantum mechanics, especially when we dealing with electron spin of Pauli theory [3, 13, 21]. Cornwell in [5] give the proof by considering the simply connectedness of $SU(2)$, especially when arguing that the homomorphism maps any elements of $SU(2)$ into $SO(3)$, rather than into $O(3)$. Donchev et.al in [6] gave the proof by using the Cayley maps for the isomorphic Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$. Sattinger and Weaver in [23] construct the homomorphism between $SU(2)$ to $SO(3)$ by creating mobius transformation for a rotation of $SO(3)$.

Nevertheless, as long as our searching in various literatures, we never found a complete explicit computation of homomorphism from $SU(2)$ to $SO(3)$ by purely algebraic ways, without concerning their topology. Motivated by this fact, in this paper we give an explicit complete purely algebraic proof of homomorphism of $SU(2)$ to $SO(3)$ without concerning their topological properties. We hope this research will give an alternatif explanation of homomorphism $SU(2)$ to $SO(3)$ without having to learn topology first.

2. Rotation in $\mathbb{R}^3$

Each element X in $\mathbb{R}^3$ can be expressed in the following form

$$X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{4}$$

where $x^1, x^2, x^3 \in \mathbb{R}$. The norm of X is defined by

$$|X|^2 = X^T X = (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{5}$$

Every $A \in SO(3)$ is a rotation transformation in $\mathbb{R}^3$ since if $X' = AX$, where $X \in \mathbb{R}^3$, then it follow that

$$\begin{aligned} |X'|^2 &= X'^T X' = (AX)^T (AX) = X^T A^T A X \\ &= X^T \mathbb{I}_3 X = X^T X = |X|^2. \end{aligned} \tag{6}$$

3. Rotation in $\mathbb{H}$

According to the definition of $\mathbb{H}$, every $V \in \mathbb{H}$ may be expressed in the following form

$$V = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}. \tag{7}$$

One of bases in the vector space $\mathbb{H}$ are the following three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

so that each vector $V \in \mathbb{H}$ may be expressed as follow

$$V = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = x^i\sigma_i. \tag{9}$$

Note that the above three Pauli matrices satisfy the following properties

$$tr(\sigma_i) = 0, \tag{10}$$

$$\sigma_i^\dagger = \sigma_i, \tag{11}$$

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I}_2, \tag{12}$$

for all $i, j, k = 1, 2, 3$, where ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 1, & (ijk) = (123), (231), (312) \\ -1, & (ijk) = (213), (132), (321) \\ 0, & \text{others} \end{cases} \tag{13}$$

and δ_{ij} is a kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{14}$$

The norm of $V \in \mathbb{H}$ is defined by

$$|V|^2 \equiv -det(V) = (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{15}$$

The rotation of V by $U \in SU(2)$ is defined by

$$V' \equiv UVU^\dagger, \tag{16}$$

since V' is hermitian matrix, that is

$$V'^\dagger = (UVU^\dagger)^\dagger = (U^\dagger)^\dagger V^\dagger U^\dagger = UVU^\dagger = V', \tag{17}$$

and traceless, that is

$$\begin{aligned} tr(V') &= tr(UVU^\dagger) = tr(VU^\dagger U) = tr(V\mathbb{I}_2) \\ &= tr(V) = 0, \end{aligned} \tag{18}$$

and also having same norm as V, that is

$$\begin{aligned} |V'|^2 &= -det(V') = -det(UVU^\dagger) \\ &= -det(U)det(V)det(U^\dagger) \\ &= -1 \cdot det(V) \cdot 1 = -det(V) = |V|^2. \end{aligned} \tag{19}$$

4. Homomorphism from $SU(2)$ to $SO(3)$

In order to find a homomorphism from $SU(2)$ to $SO(3)$, we note that $V' = x'^i\sigma_i = UVU^\dagger$ and $V = x^i\sigma_i$. Hence, by using eq.(12) we obtain

$$\begin{aligned} x'^i &= \frac{1}{2}tr(\sigma_i V') = \frac{1}{2}tr(\sigma_i UVU^\dagger) \\ &= \frac{1}{2}tr(\sigma_i U x^j \sigma_j U^\dagger). \\ &= \frac{1}{2}tr(\sigma_i U \sigma_j U^\dagger) x^j. \end{aligned} \tag{20}$$

Meanwhile we know that if a vector $X = (x^1\ x^2\ x^3)^T \in \mathbb{R}^3$ is transformed by $A \in O(3)$, then we will get a new vector, say $V' = (x'^1\ x'^2\ x'^3)^T$, according to formula

$$x'^i = [A]^i{}_j x^j. \tag{21}$$

Hence we may conclude that the entries of matrix $A \in SO(3)$ may be written in the expression of the Pauli matrices as follow

$$[A]^i{}_j = \frac{1}{2} tr(\sigma_i U \sigma_j U^\dagger). \tag{22}$$

More over, we can try to start from a map $\varphi : SU(2) \to ML(3, \mathbb{C})$ defined by

$$[\varphi(U)]^i{}_j \equiv \frac{1}{2} tr(\sigma_i U \sigma_j U^\dagger), \tag{23}$$

and then show that $[\varphi(U)]$ is in $SO(3)$ whenever $U \in SU(2)$. According to eq.22, eq.21 and eq.20, it is clear that that $\varphi(U)$ defined in eq.23 is belong to $O(3)$. However in this article we will show that $\varphi(U)$ in eq.(23) is an element of $O(3)$ by using the property of orthogonality of the elements of $O(3)$, i.e. for every $A \in O(3)$ we have

$$AA^T = A^T A = \mathbb{I}_3. \tag{24}$$

Now for first callculation we will prove that the map φ satisfy

$$[\varphi(U_1 U_2)]^i{}_j = [\varphi(U_1)]^i{}_k [\varphi(U_2)]^k{}_j, \tag{25}$$

for every $U_1, U_2 \in SU(2)$. This will provide us the homomorphism property of the maps defined in eq.(23). By using eq.(23), the right side of eq.(25) become

$$\begin{aligned} [\varphi(U_1)]^i{}_k [\varphi(U_2)]^k{}_j &= \frac{1}{2} tr(\sigma_i U_1 \sigma_k U_1^\dagger) \frac{1}{2} tr(\sigma_k U_2 \sigma_j U_2^\dagger) \\ &= \frac{1}{4} tr(\sigma_k U_1^\dagger \sigma_i U_1) tr(\sigma_k U_2 \sigma_j U_2^\dagger) \\ &= \frac{1}{4} tr(\sigma_k \Omega_{1i}) tr(\sigma_k \Omega'_{2j}), \end{aligned} \tag{26}$$

where

$$\Omega_{ki} \equiv U_k^\dagger \sigma_i U_k, \qquad \Omega'_{ki} \equiv U_k \sigma_i U_k^\dagger. \tag{27}$$

According to the definition of trace, eq.(26) become

$$\begin{aligned} [\varphi(U_1)]^i{}_k [\varphi(U_2)]^k{}_j &= \frac{1}{4} ([\sigma_k]^\alpha{}_\beta [\Omega_{1i}]^\beta{}_\alpha)([\sigma_k]^\gamma{}_\delta [\Omega'_{2j}]^\delta{}_\gamma) \\ &= \frac{1}{4} \Xi^{\alpha\gamma}{}_{\beta\delta} [\Omega_{1i}]^\beta{}_\alpha [\Omega'_{2j}]^\delta{}_\gamma, \end{aligned} \tag{28}$$

where

$$\Xi^{\alpha\gamma}{}_{\beta\delta} \equiv [\sigma_k]^\alpha{}_\beta [\sigma_k]^\gamma{}_\delta. \tag{29}$$

Note that since $\alpha, \beta, \gamma, \delta = 1, 2$ there are 16 combinations of $\alpha, \beta, \gamma, \delta$. The computation of those 16 values of $\Xi^{\alpha\gamma}{}_{\beta\delta}$ are given below :

$$\begin{aligned} \Xi^{11}{}_{11} &= [\sigma_1]^1{}_1 [\sigma_1]^1{}_1 + [\sigma_2]^1{}_1 [\sigma_2]^1{}_1 + [\sigma_3]^1{}_1 [\sigma_3]^1{}_1 \\ &= 0 + 0 + 1 = 1 \\ \Xi^{11}{}_{12} &= [\sigma_1]^1{}_1 [\sigma_1]^1{}_2 + [\sigma_2]^1{}_1 [\sigma_2]^1{}_2 + [\sigma_3]^1{}_1 [\sigma_3]^1{}_2 \\ &= 0 + 0 + 0 = 0 \\ \Xi^{11}{}_{21} &= [\sigma_1]^1{}_2 [\sigma_1]^1{}_1 + [\sigma_2]^1{}_2 [\sigma_2]^1{}_1 + [\sigma_3]^1{}_2 [\sigma_3]^1{}_1 \\ &= 0 + 0 + 0 = 0 \\ \Xi^{12}{}_{11} &= [\sigma_1]^1{}_1 [\sigma_1]^2{}_1 + [\sigma_2]^1{}_1 [\sigma_2]^2{}_1 + [\sigma_1]^1{}_1 [\sigma_3]^2{}_1 \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \Xi^{21}{}_{11} &= [\sigma_1]^2{}_1 [\sigma_1]^1{}_1 + [\sigma_2]^2{}_1 [\sigma_2]^1{}_1 + [\sigma_3]^2{}_1 [\sigma_3]^1{}_1 \\ &= 0 + 0 + 0 = 0 \\ \Xi^{11}{}_{22} &= [\sigma_1]^1{}_2 [\sigma_1]^1{}_2 + [\sigma_2]^1{}_2 [\sigma_2]^1{}_2 + [\sigma_3]^1{}_2 [\sigma_3]^1{}_2 \\ &= 1 + (-1) + 0 = 0 \\ \Xi^{12}{}_{12} &= [\sigma_1]^1{}_1 [\sigma_1]^2{}_2 + [\sigma_2]^1{}_1 [\sigma_2]^2{}_2 + [\sigma_3]^1{}_1 [\sigma_3]^2{}_2 \\ &= 0 + 0 + (-1) = -1 \\ \Xi^{21}{}_{12} &= [\sigma_1]^2{}_1 [\sigma_1]^1{}_2 + [\sigma_2]^2{}_1 [\sigma_2]^1{}_2 + [\sigma_3]^2{}_1 [\sigma_3]^1{}_2 \\ &= 1 + 1 + 0 = 2 \\ \Xi^{21}{}_{21} &= \Xi^{12}{}_{12} = -1 \\ \Xi^{22}{}_{11} &= [\sigma_1]^2{}_1 [\sigma_1]^2{}_1 + [\sigma_2]^2{}_1 [\sigma_2]^2{}_1 + [\sigma_3]^2{}_1 [\sigma_3]^2{}_1 \\ &= 1 + (-1) + 0 = 0 \\ \Xi^{12}{}_{21} &= \Xi^{21}{}_{12} = 2 \\ \Xi^{12}{}_{22} &= [\sigma_1]^1{}_2 [\sigma_1]^2{}_2 + [\sigma_2]^1{}_2 [\sigma_2]^2{}_2 + [\sigma_3]^1{}_2 [\sigma_3]^2{}_2 \\ &= 0 + 0 + 0 = 0 \\ \Xi^{21}{}_{22} &= \Xi^{12}{}_{22} = 0 \\ \Xi^{22}{}_{12} &= [\sigma_1]^2{}_1 [\sigma_1]^2{}_2 + [\sigma_2]^2{}_1 [\sigma_2]^2{}_2 + [\sigma_3]^2{}_1 [\sigma_3]^2{}_2 \\ &= 0 + 0 + 0 = 0 \\ \Xi^{22}{}_{21} &= \Xi^{22}{}_{12} = 0 \\ \Xi^{22}{}_{22} &= [\sigma_1]^2{}_2 [\sigma_1]^2{}_2 + [\sigma_2]^2 2 [\sigma_2]^2{}_2 + [\sigma_3]^2{}_2 [\sigma_3]^2{}_2 \\ &= 0 + 0 + 1 = 1. \end{aligned}$$

By using the above 16 values of $\Xi^{\alpha\gamma}{}_{\beta\delta}$, eq.(28) become

$$\begin{aligned} &[\varphi(U_1)]^i{}_k [\varphi(U_2)]^k{}_j = \\ &\quad \frac{1}{4} ([\Omega_{1i}]^1{}_1 [\Omega'_{2j}]^1{}_1 + [\Omega_{1i}]^2{}_2 [\Omega'_{2j}]^2{}_2 \\ &\quad + 2[\Omega_{1i}]^1{}_2 [\Omega'_{2j}]^2{}_1 \\ &\quad + 2[\Omega_{1i}]^2{}_1 [\Omega'_{2j}]^1{}_2 - [\Omega_{1i}]^1{}_1 [\Omega'_{2j}]^2{}_2 \\ &\quad - [\Omega_{1i}]^2{}_2 [\Omega'_{2j}]^1{}_1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}(2([\Omega_{1i}]^1{}_1[\Omega'_{2j}]^1{}_1 + [\Omega_{1i}]^2{}_2[\Omega'_{2j}]^2{}_2 \\
&\quad + [\Omega_{1i}]^1{}_2[\Omega'_{2j}]^2{}_1 + [\Omega_{1i}]^2{}_1[\Omega'_{2j}]^1{}_2) \\
&\quad - ([\Omega_{1i}]^1{}_1[\Omega'_{2j}]^1{}_1 + [\Omega_{1i}]^2{}_2[\Omega'_{2j}]^2{}_2 \\
&\quad + [\Omega_{1i}]^1{}_1[\Omega'_{2j}]^2{}_2 + [\Omega_{1i}]^2{}_2[\Omega'_{2j}]^1{}_1)) \\
&= \frac{1}{4}(2 \cdot tr([\Omega_{1i}][\Omega'_{2j}]) \\
&\quad - ([\Omega_{1i}]^1{}_1[\Omega'_{2j}]^1{}_1 + [\Omega_{1i}]^2{}_2[\Omega'_{2j}]^2{}_2 \\
&\quad + [\Omega_{1i}]^1{}_1[\Omega'_{2j}]^2{}_2 + [\Omega_{1i}]^2{}_2[\Omega'_{2j}]^1{}_1)) \\
&= \frac{1}{4}(2 \cdot tr([\Omega_{1i}][\Omega'_{2j}]) \\
&\quad - ([\Omega_{1i}]^1{}_1([\Omega'_{2j}]^1{}_1 + [\Omega'_{2j}]^2{}_2) \\
&\quad + [\Omega_{1i}]^2{}_2([\Omega'_{2j}]^1{}_1 + [\Omega'_{2j}]^2{}_2))) \\
&= \frac{1}{4}(2 \cdot tr([\Omega_{1i}][\Omega'_{2j}]) \\
&\quad - (([\Omega_{1i}]^1{}_1 + [\Omega_{1i}]^2{}_2)([\Omega'_{2j}]^1{}_1 + [\Omega'_{2j}]^2{}_2))) \\
&= \frac{1}{4}(2 \cdot tr([\Omega_{1i}][\Omega'_{2j}]) - (tr(\Omega_{1i})tr(\Omega'_{2j}))) \qquad (30)
\end{aligned}
$$

According to the definition given in (27) and the properties of σ_i given in eq.(11) and (10) then Ω_{ki} and Ω'_{ki} are hermitian matrices, because

$$
\begin{aligned}
\Omega^\dagger_{ki} &= (U_k\sigma_i U_k^\dagger)^\dagger = U_k\sigma_i U_k^\dagger = \Omega_{ki}, \\
\Omega'^\dagger_{ki} &= (U_k^\dagger\sigma_i U_k)^\dagger = U_k^\dagger\sigma_i U_k = \Omega'_{ki},
\end{aligned} \qquad (31)
$$

and they are also traceless because

$$
\begin{aligned}
tr(\Omega_{ki}) &= tr(U_k\sigma_i U_k^\dagger) = tr(U_k^\dagger U_k\sigma_i) = tr(\sigma_i) = 0, \\
tr(\Omega'_{ki}) &= tr(U_k^\dagger\sigma_i U_k) = tr(U_k U_k^\dagger\sigma_i) = tr(\sigma_i) = 0.
\end{aligned} \qquad (32)
$$

By using the two properties of Ω_{ki} and Ω'_{ki} above, eq.(30) may be written as follows

$$
\begin{aligned}
[\varphi(U_1)]^i{}_k[\varphi(U_2)]^k{}_j &= \frac{1}{4}(2 \cdot tr([\Omega_{1i}][\Omega'_{2j}]) \\
&\quad - (tr(\Omega_{1i})tr(\Omega'_{2j}))) \\
&= \frac{1}{4}(2 \cdot tr([\Omega_{1i}][\Omega'_{2j}]) - 0 \cdot 0) \\
&= \frac{1}{2}tr([\Omega_{1i}][\Omega'_{2j}]).
\end{aligned} \qquad (33)
$$

Finally, according to the definition of Ω_{ki} and Ω'_{ki}, we get

$$
\begin{aligned}
[\varphi(U_1)]^i{}_k[\varphi(U_2)]^k{}_j &= \frac{1}{2}tr([\Omega_{1i}][\Omega'_{2j}]) \\
&= \frac{1}{2}tr(U_1^\dagger\sigma_i U_1 U_2\sigma_j U_2^\dagger) \\
&= \frac{1}{2}tr(\sigma_i U_1 U_2\sigma_j U_2^\dagger U_1^\dagger) \\
&= \frac{1}{2}tr(\sigma_i (U_1 U_2)\sigma_j (U_1 U_2)^\dagger) \\
&= [\varphi(U_1U_2)]^i{}_j.
\end{aligned} \qquad (34)
$$

Next we will prove the following two conditions

$$
\varphi(U)\varphi(U)^T = \mathbb{I}_3 \quad \text{and} \quad det(\varphi(U)) = 1. \qquad (35)
$$

for all $U \in SU(2)$. The first condition is needed to ensure that the collections of $\varphi(U)$ is having orthogonal property and the second condition is needed to ensure that $\varphi(U)$ is belong to $SL(3,\mathbb{R})$, for every $U \in SU(2)$. The first condition may be written in the expression of matrix entries as follows

$$
[\varphi(U)]^i{}_k[\varphi(U)^T]^k{}_j = \delta_{ij}. \qquad (36)
$$

Since

$$
[\varphi(U)^T]^k{}_j = [\varphi(U)]^j{}_k = \frac{1}{2}tr(\sigma_j U\sigma_k U^\dagger), \qquad (37)
$$

then the left side of eq.(36) become

$$
\begin{aligned}
[\varphi(U)]^i{}_k[\varphi(U)^T]^k{}_j &= \frac{1}{2}tr(\sigma_i U\sigma_k U^\dagger)\frac{1}{2}tr(\sigma_j U\sigma_k U^\dagger) \\
&= \frac{1}{4}tr(\sigma_k U^\dagger\sigma_i U)tr(\sigma_k U^\dagger\sigma_j U) \\
&= \frac{1}{4}tr(\sigma_k\Omega_i)tr(\sigma_k\Omega_j).
\end{aligned} \qquad (38)
$$

By doing the similar computation as was done from eq.(28) until eq.(33), then the last equation become

$$
\begin{aligned}
[\varphi(U)]^i{}_k[\varphi(U)^T]^k{}_j &= \frac{1}{2}tr(\Omega_i\Omega_j) \\
&= \frac{1}{2}tr(U^\dagger\sigma_i U U^\dagger\sigma_j U) \\
&= \frac{1}{2}tr(U^\dagger\sigma_i\sigma_j U) \\
&= \frac{1}{2}tr(UU^\dagger\sigma_i\sigma_j) = \frac{1}{2}tr(\sigma_i\sigma_j).
\end{aligned} \qquad (39)
$$

Next, by using eq.(12), eq.(39) become

$$
\begin{aligned}
[\varphi(U)]^i{}_k[\varphi(U)^T]^k{}_j &= \frac{1}{2}tr(i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I}_2) \\
&= i\epsilon_{ijk}\frac{1}{2}tr(\sigma_k) + \frac{1}{2}\delta_{ij}tr(\mathbb{I}_2).
\end{aligned} \qquad (40)
$$

However according to eq.(10) and $tr(\mathbb{I}_2) = 2$, we obtain

$$[\varphi(U)]^i{}_k[\varphi(U)^T]^k{}_j = \delta_{ij}, \quad (41)$$

that is $\varphi(U) \in O(3)$, $\forall U \in SU(2)$.

For the second condition in eq.(35), according to the definition of determinant of a matrix, $det(\varphi(U))$ may be written in the following form

$$det(\varphi(U)) = \epsilon^{ijk}[\varphi(U)]^1{}_i[\varphi(U)]^2{}_j[\varphi(U)]^3{}_k. \quad (42)$$

Using the definition of $\varphi(U)$, then we have

$$\begin{aligned} det(\varphi(U)) &= \epsilon^{ijk}\frac{1}{2}tr(\sigma_1 U\sigma_i U^\dagger)\frac{1}{2}tr(\sigma_2 U\sigma_j U^\dagger) \\ &\quad \times \frac{1}{2}tr(\sigma_3 U\sigma_k U^\dagger) \\ &= \frac{1}{8}\epsilon^{ijk}tr(\sigma_1\Omega'_i)tr(\sigma_1\Omega'_j)tr(\sigma_1\Omega'_k) \\ &= \frac{1}{8}\epsilon^{ijk}([\sigma_1]^\alpha{}_\beta[\Omega'_i]^\beta{}_\alpha)([\sigma_2]^\gamma{}_\delta[\Omega'_j]^\delta{}_\gamma) \\ &\quad \times ([\sigma_3]^\mu{}_\nu[\Omega'_k]^\nu{}_\mu) \\ &= \frac{1}{8}\epsilon^{ijk}\Gamma^{\alpha\gamma\mu}{}_{\beta\delta\nu}[\Omega'_i]^\beta{}_\alpha[\Omega'_j]^\delta{}_\gamma[\Omega'_k]^\nu{}_\mu, \end{aligned} \quad (43)$$

where

$$\Gamma^{\alpha\gamma\mu}{}_{\beta\delta\nu} \equiv [\sigma_1]^\alpha{}_\beta[\sigma_2]^\gamma{}_\delta[\sigma_3]^\mu{}_\nu. \quad (44)$$

There are only 8 combinations of $\alpha, \delta, \mu, \beta, \delta, \nu$ having non zero values, that is

$$\begin{aligned} \Gamma^{111}{}_{221} &= [\sigma_1]^1{}_2[\sigma_2]^1{}_2[\sigma_3]^1{}_1 = 1\cdot(-i)\cdot 1 = -i, \\ \Gamma^{112}{}_{222} &= [\sigma_1]^1{}_2[\sigma_2]^1{}_2[\sigma_3]^2{}_2 = 1\cdot(-i)\cdot(-1) = i, \\ \Gamma^{121}{}_{211} &= [\sigma_1]^1{}_2[\sigma_2]^2{}_1[\sigma_3]^1{}_1 = 1\cdot(i)\cdot 1 = i, \\ \Gamma^{122}{}_{212} &= [\sigma_1]^1{}_2[\sigma_2]^2{}_1[\sigma_3]^2{}_2 = 1\cdot(i)\cdot(-1) = -i, \\ \Gamma^{211}{}_{121} &= [\sigma_1]^2{}_1[\sigma_2]^1{}_2[\sigma_3]^1{}_1 = 1\cdot(-i)\cdot 1 = -i, \\ \Gamma^{212}{}_{122} &= [\sigma_1]^2{}_1[\sigma_2]^1{}_2[\sigma_3]^2{}_2 = 1\cdot(-i)\cdot(-1) = i, \\ \Gamma^{221}{}_{111} &= [\sigma_1]^2{}_1[\sigma_2]^2{}_1[\sigma_3]^1{}_1 = 1\cdot(i)\cdot 1 = i, \\ \Gamma^{222}{}_{112} &= [\sigma_1]^2{}_1[\sigma_2]^2{}_1[\sigma_3]^2{}_2 = 1\cdot(i)\cdot(-1) = -i. \end{aligned} \quad (45)$$

Now eq.(43) become

$$\begin{aligned} det(\varphi(U)) = \frac{1}{8}\epsilon^{ijk}(&-i[\Omega'_i]^2{}_1[\Omega'_j]^2{}_1[\Omega'_k]^1{}_1 \\ &+ i[\Omega'_i]^2{}_1[\Omega'_j]^2{}_1[\Omega'_k]^2{}_2 \\ &+ i[\Omega'_i]^2{}_1[\Omega'_j]^1{}_2[\Omega'_k]^1{}_1 \\ &- i[\Omega'_i]^2{}_1[\Omega'_j]^1{}_2[\Omega'_k]^2{}_2 \\ &- i[\Omega'_i]^1{}_2[\Omega'_j]^2{}_1[\Omega'_k]^1{}_1 \\ &+ i[\Omega'_i]^1{}_2[\Omega'_j]^2{}_1[\Omega'_k]^2{}_2 \\ &+ i[\Omega'_i]^1{}_2[\Omega'_j]^1{}_2[\Omega'_k]^1{}_1 \\ &- i[\Omega'_i]^1{}_2[\Omega'_j]^1{}_2[\Omega'_k]^2{}_2) \end{aligned} \quad (46)$$

By arranging the term we obtain

$$\begin{aligned} det(\varphi(U)) &= \frac{1}{8}\epsilon^{ijk}(-i([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)[\Omega'_j]^2{}_1[\Omega'_k]^1{}_1 \\ &\quad + i([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)[\Omega'_j]^2{}_1[\Omega'_k]^2{}_2 \\ &\quad + i([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)[\Omega'_j]^1{}_2[\Omega'_k]^1{}_1 \\ &\quad - i([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)[\Omega'_j]^1{}_2[\Omega'_k]^2{}_2) \\ &= \frac{1}{8}\epsilon^{ijk}(i([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)[\Omega'_j]^2{}_1([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1) \\ &\quad - i([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)[\Omega'_j]^1{}_2([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1)) \\ &= \frac{1}{8}i\epsilon^{ijk}([\Omega'_i]^2{}_1 + [\Omega'_i]^1{}_2)([\Omega'_j]^2{}_1 \\ &\quad - [\Omega'_j]^1{}_2)([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1) \\ &= \frac{1}{8}i\epsilon^{ijk}([\Omega'_i]^2{}_1[\Omega'_j]^2{}_1([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1) \\ &\qquad - [\Omega'_i]^1{}_2[\Omega'_j]^1{}_2([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1) \\ &\qquad + [\Omega'_i]^1{}_2[\Omega'_j]^2{}_1([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1) \\ &\qquad - [\Omega'_i]^2{}_1[\Omega'_j]^1{}_2([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1)) \\ &= \frac{1}{8}i\epsilon^{ijk}([\Omega'_i]^1{}_2[\Omega'_j]^2{}_1 - [\Omega'_i]^2{}_1[\Omega'_j]^1{}_2) \\ &\quad \times ([\Omega'_k]^2{}_2 - [\Omega'_k]^1{}_1) \end{aligned} \quad (47)$$

$$\begin{aligned} &= \frac{1}{8}i\epsilon^{ijk}(2i\,\mathrm{Im}([\Omega'_i]^1{}_2[\Omega'_j]^2{}_1))(-2[\Omega'_k]^1{}_1) \\ &= \frac{1}{2}\epsilon^{ijk}\mathrm{Im}([\Omega'_i]^1{}_2[\Omega'_j]^2{}_1)[\Omega'_k]^1{}_1 \\ &= \frac{1}{2}\Big[\epsilon^{123}\mathrm{Im}([\Omega'_1]^1{}_2[\Omega'_2]^2{}_1)[\Omega'_3]^1{}_1 \\ &\quad + \epsilon^{213}\mathrm{Im}([\Omega'_2]^1{}_2[\Omega'_1]^2{}_1)[\Omega'_3]^1{}_1 \\ &\quad + \epsilon^{312}\mathrm{Im}([\Omega'_3]^1{}_2[\Omega'_1]^2{}_1)[\Omega'_2]^1{}_1 \\ &\quad + \epsilon^{132}\mathrm{Im}([\Omega'_1]^1{}_2[\Omega'_3]^2{}_1)[\Omega'_2]^1{}_1 \\ &\quad + \epsilon^{231}\mathrm{Im}([\Omega'_2]^1{}_2[\Omega'_3]^2{}_1)[\Omega'_1]^1{}_1 \\ &\quad + \epsilon^{321}\mathrm{Im}([\Omega'_3]^1{}_2[\Omega'_2]^2{}_1)[\Omega'_1]^1{}_1\Big] \\ &= \frac{1}{2}\Big[\Big(\mathrm{Im}([\Omega'_1]^1{}_2[\Omega'_2]^2{}_1) - \mathrm{Im}([\Omega'_2]^1{}_2[\Omega'_1]^2{}_1)\Big) \\ &\quad \times [\Omega'_3]^1{}_1 \\ &\quad + \Big(\mathrm{Im}([\Omega'_3]^1{}_2[\Omega'_1]^2{}_1) - \mathrm{Im}([\Omega'_1]^1{}_2[\Omega'_3]^2{}_1)\Big) \\ &\quad \times [\Omega'_2]^1{}_1 \\ &\quad + \Big(\mathrm{Im}([\Omega'_2]^1{}_2[\Omega'_3]^2{}_1) - \mathrm{Im}([\Omega'_3]^1{}_2[\Omega'_2]^2{}_1)\Big) \\ &\quad \times [\Omega'_1]^1{}_1\Big] \end{aligned} \quad (48)$$

The values of $[\Omega'_i]^1{}_2$, $[\Omega'_j]^2{}_1$, and $[\Omega'_k]^1{}_1$, for $i, j, k =$

1, 2, 3, are given below

$$\begin{aligned}[\Omega'_1]^1{}_2 &= [U\sigma_1 U^\dagger]^1{}_2 \\ &= (U)^1{}_1(\sigma_1)^1{}_2(U^\dagger)^2{}_2 + (U)^1{}_2(\sigma_1)^2{}_1(U^\dagger)^1{}_2 \\ &= \cos\theta e^{i\zeta}\cos\theta e^{i\zeta} - \sin\theta e^{i\eta}\sin\theta e^{i\eta} \\ &= \cos^2\theta e^{2i\zeta} - \sin^2\theta e^{2i\eta}\end{aligned} \tag{49}$$

$$\begin{aligned}[\Omega'_1]^2{}_1 &= [U\sigma_1 U^\dagger]^2{}_1 \\ &= (U)^2{}_1(\sigma_1)^1{}_2(U^\dagger)^2{}_1 + (U)^2{}_2(\sigma_1)^2{}_1(U^\dagger)^1{}_1 \\ &= \sin\theta e^{-i\eta}(-\sin\theta e^{-i\eta}) + \cos\theta e^{-i\zeta}\cos\theta e^{-i\zeta} \\ &= -\sin^2\theta e^{-2i\eta} + \cos^2\theta e^{-2i\zeta}\end{aligned} \tag{50}$$

$$\begin{aligned}[\Omega'_1]^1{}_1 &= [U\sigma_1 U^\dagger]^1{}_1 \\ &= (U)^1{}_2(\sigma_1)^2{}_1(U^\dagger)^1{}_1 + (U)^1{}_1(\sigma_1)^1{}_2(U^\dagger)^2{}_1 \\ &= -\sin\theta e^{i\eta}\cos\theta e^{-i\zeta} - \cos\theta e^{i\zeta}\sin\theta e^{-i\eta} \\ &= -\cos\theta\sin\theta(e^{-i(\zeta-\eta)} + e^{i(\zeta-\eta)}) \\ &= -2\cos\theta\sin\theta\cos(\zeta-\eta)\end{aligned} \tag{51}$$

$$\begin{aligned}[\Omega'_2]^1{}_2 &= [U\sigma_2 U^\dagger]^1{}_2 \\ &= (U)^1{}_2(\sigma_2)^2{}_1(U^\dagger)^1{}_2 + (U)^1{}_1(\sigma_2)^1{}_2(U^\dagger)^2{}_2 \\ &= -\sin\theta e^{i\eta}(i)\sin\theta e^{i\eta} + \cos\theta e^{i\zeta}(-i)\cos\theta e^{i\zeta} \\ &= -i(\sin^2\theta e^{2i\eta} + \cos^2\theta e^{2i\zeta})\end{aligned} \tag{52}$$

$$\begin{aligned}[\Omega'_2]^2{}_1 &= [U\sigma_2 U^\dagger]^2{}_1 \\ &= (U)^2{}_1(\sigma_2)^1{}_2(U^\dagger)^2{}_1 + (U)^2{}_2(\sigma_2)^2{}_1(U^\dagger)^1{}_1 \\ &= \sin\theta e^{-i\eta}(-i) - \sin\theta e^{-i\eta} \\ &\quad + \cos\theta e^{-i\zeta}(i)\cos\theta e^{-i\zeta} \\ &= i(\sin^2\theta e^{-2i\eta} + \cos^2\theta e^{-2i\zeta})\end{aligned} \tag{53}$$

$$\begin{aligned}[\Omega'_2]^1{}_1 &= [U\sigma_2 U^\dagger]^1{}_1 \\ &= (U)^1{}_1(\sigma_2)^1{}_2(U^\dagger)^2{}_1 + (U)^1{}_2(\sigma_2)^2{}_1(U^\dagger)^1{}_1 \\ &= \cos\theta e^{i\zeta}(-i)(-\sin\theta e^{i\eta}) \\ &\quad - \sin\theta e^{i\eta}(i)\cos\theta e^{-i\zeta} \\ &= i\cos\theta\sin\theta(e^{i(\zeta-\eta)} - e^{-i(\zeta-\eta)}) \\ &= i\cos\theta\sin\theta(2i\sin(\zeta-\eta)) \\ &= -2\cos\theta\sin\theta\sin(\zeta-\eta)\end{aligned} \tag{54}$$

$$\begin{aligned}[\Omega'_3]^1{}_2 &= [U\sigma_3 U^\dagger]^1{}_2 \\ &= (U)^1{}_1(\sigma_3)^1{}_1(U^\dagger)^1{}_2 + (U^1)_2(\sigma_3)^2{}_2(U^\dagger)^2{}_2 \\ &= \cos\theta e^{i\zeta}\sin\theta e^{i\eta} + \cos\theta e^{i\zeta}\sin\theta e^{i\eta} \\ &= 2\cos\theta\sin\theta e^{i(\zeta+\eta)}\end{aligned} \tag{55}$$

$$\begin{aligned}[\Omega'_3]^2{}_1 &= [U\sigma_3 U^\dagger]^2{}_1 \\ &= (U)^2{}_1(\sigma_3)^1{}_1(U^\dagger)^1{}_1 + (U)^2{}_2(\sigma_3)^2{}_2(U^\dagger)^2{}_1 \\ &= \sin\theta e^{-i\eta}\cos\theta e^{-i\zeta} + \cos\theta e^{-i\zeta}\sin\theta e^{-i\eta} \\ &= 2\cos\theta\sin\theta e^{-i(\zeta+\eta)}\end{aligned} \tag{56}$$

$$\begin{aligned}[\Omega'_3]^1{}_1 &= [U\sigma_3 U^\dagger]^1{}_1 \\ &= (U)^1{}_1(\sigma_3)^1{}_1(U^\dagger)^1{}_1 + (U)^1{}_2(\sigma_3)^2{}_2(U^\dagger)^2{}_1 \\ &= \cos\theta e^{i\zeta}\cos\theta e^{-i\zeta} + (-\sin\theta e^{i\eta})(-1)(-\sin\theta e^{-i\eta}) \\ &= \cos^2\theta - \sin^2\theta\end{aligned} \tag{57}$$

Now, we can compute the values of $\mathrm{Im}([\Omega'_i]^1{}_2[\Omega'_j]^2{}_1$, for all $i, j = 1, 2, 3$, as follows

$$\begin{aligned}\mathrm{Im}([\Omega'_1]^1{}_2[\Omega'_2]^2{}_1) &= \mathrm{Im}((\cos^2\theta e^{2i\zeta} - \sin^2\theta e^{2i\eta}) \\ &\quad \times (i\cos^2\theta e^{-2i\zeta} + i\sin^2\theta e^{-2i\eta})) \\ &= \mathrm{Im}(i(\cos^4\theta - \sin^4\theta \\ &\quad + \cos^2\theta\sin^2\theta e^{2i(\zeta-\eta)} \\ &\quad - \cos^2\theta\sin^2\theta e^{-2i(\zeta-\eta)})) \\ &= \mathrm{Im}(i(\cos^4\theta - \sin^4\theta \\ &\quad + \cos^2\theta\sin^2\theta(\cos 2(\zeta-\eta) \\ &\quad + i\sin 2(\zeta-\eta)))) \\ &\quad - \cos^2\theta\sin^2\theta(\cos 2(\zeta-\eta) \\ &\quad - i\sin 2(\zeta-\eta)))) \\ &= \cos^4\theta - \sin^4\theta\end{aligned} \tag{58}$$

$$\begin{aligned}\mathrm{Im}([\Omega_2']^1{}_2[\Omega_1']^2{}_1) &= \mathrm{Im}((-i\sin^2\theta e^{2i\eta} - i\cos^2\theta e^{2i\zeta})\\ &\quad \times(-\sin^2\theta e^{-2i\eta} + \cos^2\theta e^{-2i\zeta}))\\ &= \mathrm{Im}(i(\sin^4\theta - \cos^4\theta\\ &\quad + \cos^2\theta\sin^2\theta e^{2i(\zeta-\eta)}\\ &\quad - \cos^2\theta\sin^2\theta e^{-2i(\zeta-\eta)}))\\ &= \mathrm{Im}(i(\sin^4\theta - \cos^4\theta\\ &\quad + \cos^2\theta\sin^2\theta(\cos 2(\zeta-\eta)\\ &\quad + i\sin 2(\zeta-\eta))\\ &\quad - \cos^2\theta\sin^2\theta(\cos 2(\zeta-\eta)\\ &\quad - i\sin 2(\zeta-\eta))))\\ &= \mathrm{Im}(i(\sin^4\theta - \cos^4\theta\\ &\quad + 2i\cos^2\theta\sin^2\theta\sin 2(\zeta-\eta)))\\ &= \sin^4\theta - \cos^4\theta\end{aligned} \tag{59}$$

$$\begin{aligned}\mathrm{Im}([\Omega_3']^1{}_2[\Omega_1']^2{}_1) &= \mathrm{Im}((2\cos\theta e^{i\zeta}\sin\theta e^{i\eta})\\ &\quad \times(-\sin^2\theta e^{-2i\eta} + \cos^2\theta e^{-2i\zeta}))\\ &= 2\mathrm{Im}(\cos^3\theta\sin\theta e^{-i(\zeta-\eta)}\\ &\quad - \sin^3\theta\cos\theta e^{i(\zeta-\eta)})\\ &= 2\mathrm{Im}(\cos^3\theta\sin\theta(\cos(\zeta-\eta)\\ &\quad - i\sin(\zeta-\eta))\\ &\quad - \sin^3\theta\cos\theta(\cos(\zeta-\eta)\\ &\quad + i\sin(\zeta-\eta)))\\ &= -2(\cos^3\theta\sin\theta + \sin^3\theta\cos\theta)\\ &\quad \times\sin(\zeta-\eta)\end{aligned} \tag{60}$$

$$\begin{aligned}\mathrm{Im}([\Omega_1']^1{}_2[\Omega_3']^2{}_1) &= \mathrm{Im}((\cos^2\theta e^{2i\zeta} - \sin^2\theta e^{2i\eta})\\ &\quad \times(2\cos\theta e^{-i\zeta}\sin\theta e^{-i\eta}))\\ &= 2\mathrm{Im}(\cos^3\theta\sin\theta e^{i(\zeta-\eta)}\\ &\quad - \sin^3\theta\cos\theta e^{-i(\zeta-\eta)})\\ &= 2\mathrm{Im}(\cos^3\theta\sin\theta(\cos(\zeta-\eta)\\ &\quad + i\sin(\zeta-\eta))\\ &\quad - \sin^3\theta\cos\theta(\cos(\zeta-\eta)\\ &\quad - i\sin(\zeta-\eta)))\\ &= (\cos^3\theta\sin\theta + \sin^3\theta\cos\theta)\\ &\quad \times\sin(\zeta-\eta)\end{aligned} \tag{61}$$

$$\begin{aligned}\mathrm{Im}([\Omega_2']^1{}_2[\Omega_3']^2{}_1) &= \mathrm{Im}((-i\sin^2\theta e^{2i\eta} - i\cos^2\theta e^{2i\zeta})\\ &\quad \times(2\cos\theta e^{-i\zeta}\sin\theta e^{-i\eta}))\\ &= -2\mathrm{Im}(i(\sin^3\theta\cos\theta e^{-i(\zeta-\eta)}\\ &\quad + \cos^3\theta\sin\theta e^{i(\zeta-\eta)}))\\ &= -2\mathrm{Im}(i(\sin^3\theta\cos\theta(\cos(\zeta-\eta)\\ &\quad - i\sin(\zeta-\eta))\\ &\quad + \cos^3\theta\sin\theta(\cos(\zeta-\eta)\\ &\quad + i\sin(\zeta-\eta))))\\ &= -2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)\\ &\quad \times\cos(\zeta-\eta)\end{aligned} \tag{62}$$

$$\begin{aligned}\mathrm{Im}([\Omega_3']^1{}_2[\Omega_2']^2{}_1) &= \mathrm{Im}((2\cos\theta e^{i\zeta}\sin\theta e^{i\eta})\\ &\quad \times(i\sin^2\theta e^{-2i\eta} + i\cos^2\theta e^{-2i\zeta}))\\ &= 2\mathrm{Im}(i(\sin^3\theta\cos\theta e^{i(\zeta-\eta)}\\ &\quad + \cos^3\theta\sin\theta e^{-i(\zeta-\eta)}))\\ &= 2\mathrm{Im}(i(\sin^3\theta\cos\theta(\cos(\zeta-\eta)\\ &\quad + i\sin(\zeta-\eta)))\\ &\quad + \cos^3\theta\sin\theta(\cos(\zeta-\eta)\\ &\quad - i\sin(\zeta-\eta))\\ &= 2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)\\ &\quad \times\cos(\zeta-\eta)\end{aligned} \tag{63}$$

Finally, eq.(47) become

$$\begin{aligned}det(\varphi(U)) = \frac{1}{2}\Big[&((\cos^4\theta - \sin^4\theta)\\ &- (-\cos^4\theta - \sin^4\theta))(\cos^2\theta - \sin^2\theta)\\ &+ (-2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)\\ &\times\cos(\zeta-\eta))\\ &- 2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)\end{aligned} \tag{64}$$

$$
\begin{aligned}
&\times \cos(\zeta-\eta)(-2\cos\theta\sin\theta\cos(\zeta-\eta)) \\
&+ ((-2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \\
&\sin(\zeta-\eta)) \\
&- 2(\sin^3\theta\cos\theta \\
&+ \cos^3\theta\sin\theta)\sin(\zeta-\eta)) \\
&\times (-2\cos\theta\sin\theta\sin(\zeta-\eta))\Big] \\
&+ 8(\sin\theta\cos\theta)(\sin^2\theta+\cos^2\theta) \\
&\times (\cos\theta\sin\theta)] \qquad (65) \\
&= \frac{1}{2}[2(\cos^2\theta-\sin^2\theta)^2 + 8\sin^2\cos^2\theta] \\
&= \frac{1}{2}[2\cos^4\theta - 4\cos^2\theta\sin^2\theta + 2\sin^4\theta \\
&+ 8\sin^2\theta\cos^2\theta] \\
&= \frac{1}{2}[2\cos^4\theta + 4\cos^2\theta\sin^2\theta + 2\sin^4\theta] \\
&= \frac{1}{2}[2(\cos^2\theta+\sin^2\theta)^2] = \frac{1}{2}\cdot 2 = 1.
\end{aligned}
$$

Of course we have

$$
\begin{aligned}
[\varphi(\mathbb{I}_2)]^i_j &= \frac{1}{2}tr(\sigma_i\mathbb{I}_2\sigma_j\mathbb{I}_2^\dagger) = \frac{1}{2}tr(\sigma_i\sigma_j) \\
&= \frac{1}{2}tr(i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I}_2) \\
&= \frac{1}{2}(i\epsilon_{ijk}tr(\sigma_k) + \delta_{ij}tr(\mathbb{I}_2)) \\
&= \frac{1}{2}(0+2\delta_{ij}) = \delta_{ij},
\end{aligned}
$$

so we can conclude that $\varphi(\mathbb{I}_2) = \mathbb{I}_3$.

These result shows us $\varphi(U)$ is in $SO(3)$ for every U in $SU(2)$. Finnaly by using the result obtained in eq.(34), we concluded that map φ defined in eq.(23) is a homomorphism of $SU(2)$ to $SO(3)$. So, instead of considering the topological properties as in [5], we have proved by purely algebraically that the maps defined in eq.(23) will maps any elements of $SU(2)$ into $SO(3)$. Moreover, according to definition (23), it follows that

$$
\begin{aligned}
[\varphi(-U)]^i{}_j &\equiv \frac{1}{2}tr(\sigma_i(-U)\sigma_j(-U)^\dagger) \\
&= [\varphi(U)]^i{}_j \equiv \frac{1}{2}tr(\sigma_i U\sigma_j U^\dagger) \qquad (66) \\
&= [\varphi(U)]^i{}_j,
\end{aligned}
$$

so we obtain that $\varphi(-U) = \varphi(U)$, $\forall U \in SU(2)$.

5. Conclusions

The complete purely algebraic proof of homomorphism between two rotation groups, $SU(2)$ and $SO(3)$, was given by introducing a map $\varphi : SU(2) \to SO(3)$ defined as $[\varphi(U)]^i{}_j \equiv \frac{1}{2}tr(\sigma_i U\sigma_j U^\dagger)$. The proof was obtained succesfully by doing algebraic calculation, without concerning the topology of both groups.

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