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Classification of types A and A_+ from low dimensional standard and non-standard filiform Lie Algebras

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ABSTRACT
In this paper, we study low dimensional filiform Lie algebras. Specifically, three-dimensional standard filiform Lie algebras and five-dimensional non-standard filiform Lie algebras. The classification method was given in the following stage. For given a low-dimensional filiform Lie algebra, we compute its second center. We showed that three-dimensional filiform Lie algebra-called Heisenberg Lie algebra-is type A and A_+ as well. On the other hand, for $n \geq 3$, the standard filiform Lie algebras are type A but not type A_+. In this case, we give concrete example for case five-dimensional Heisenberg Lie algebra. Moreover, we proved that five-dimensional non-standard filiform Lie algebra is type A but not type A_+. It is still an open problem to classify types A and A_+ for general case of non-standard filiform Lie algebra of dimension ≥ 6.
Keywords:
Types A and A_+; Filiform Lie algebras; Heisenberg Lie algebras.
Subject Classification Code: 15B30, 17B05

Introduction

The filiform Lie algebras are interesting research area to study since they are special classes of nilpotent Lie algebras. It is well known that nilpotent Lie algebra class is perfect model to study representation theory of Lie algebras. The famous example of filiform Lie algebra is three-dimensional Heisenberg Lie algebra which is nilpotent Lie algebra as well. Kirillov built a representation theory of Lie groups started from this three-dimensional Lie algebra. Humphreys introduced representation theory of Lie algebras started from three-dimensional Heisenberg Lie algebras. This is the significance why filiform Lie algebra is important to study.

Roughly speaking, a p-dimensional nilpotent Lie algebra is called a filiform Lie algebra if its nilindex equal to $p-1$. As mentioned before, filiform Lie algebras are nilpotent Lie algebras. In addition, filiform Lie algebras form the largest class of nilpotent Lie algebras. Thus, this property is interesting to study in more comprehensive investigation. In this paper, the study starts from low- dimensional filiform Lie algebras in order to attract young researchers.

Filiform Lie algebras were studied by many researchers (Adimi & Makhlouf, 2013; de Jesus & Schneider, 2023; Falcón et al., 2016; Gómez et al., 1998). The studies of filiform Lie algebras for examples focus on their classification, index of graded filiform, center and invariants. Adimi and Makhlouf investigated of index of graded filiform Lie algebras(Adimi & Makhlouf, 2013). They classified low-dimensional filiform Lie algebras up to seven-dimensional. In this research, we concern to case of n-dimensional standard filiform Lie algebras and five-dimensional non-standard filiform Lie algebras. On the other hand, Ingrid and Daniel encouraged the notion of Heisenberg Lie algebras of type A and A_+ (Beltiţă & Beltiţă, 2015). The results in (Beltiţă & Beltiţă, 2015) was development of Sharpening of Kirillov's lemma of nilpotent Lie algebra (Kirillov, 1962).

One of the nice properties of filiform Lie algebras is generalization to Frobenius Lie algebras. It is also well known that a filiform Lie algebra is not a Frobenius Lie algebra (Alvarez et al., 2018a; Csikós & Verhóczki, 2007; Diatta & Manga, 2014; Gerstenhaber & Giaquinto, 2009; Ooms, 2009; Pham, 2016). However, filiform Lie algebras can be extended to be Frobenius Lie algebras by construction of a direct sum between a filiform Lie algebra and its split torus (Kurniadi et al., 2021; Ooms, 2009). A given nilpotent Lie algebra was classified by de Graaf (de Graaf, 2007) and a concrete computation of a split torus was given by Ayala (Ayala et al., 2012). In this paper, we shall give another property of filiform Lie algebras by classifying them for type A and A_+. We follow Ingrid and Daniel's work in case of nilpotent Lie algebras.

Given an n- standard filiform Lie algebra, applying (Beltiţă & Beltiţă, 2015), we compute its first center and second center to determine its filiform Lie algebra whether type A or/and type A_+. The method apply to case of five-dimensional non-standard filiform Lie algebras as well. Indeed, the first step, we apply center computations for three-dimensional standard filiform Lie algebra which nothing but three-dimensional Heisenberg Lie algebra. It is easy that three-dimensional Heisenberg Lie algebra is two-step nilpotent Lie algebra. The next computations, we apply for n-dimensional standard filiform Lie algebra and five-dimensional non-standard filiform Lie algebra.

The research is very important because it contributes in another type of a Lie algebra, namely we know the notion of Frobenius Lie algebras (Diatta et al., 2020; Henti et al., 2021; Ooms, 2009) and contact Lie algebras (Alvarez et al., 2016, 2018b, 2018a; Kruglikov, 1997; Rodríguez-Vallarte & Salgado, 2016; Salgado-González, 2019). Both Lie algebras are interesting to study because they can be applied to another field such as differential equations and the Classical Yang Baxter Equation.

This paper is organized as follows: in section 2, we provide the basic definitions and theorems that will be useful in obtaining our main results. We recall the notion of nilpotent and filiform Lie algebras which start from Lie algebra series, Lie algebra center: first center and second center, type A and type A_+ of nilpotent Lie algebras. In section 3, we classify filiform Lie algebras to type A and A_+. In section 4, we discuss for future research and conclusion.

Methods

We start form low dimensional filiform Lie algebra both standard and non-standard form. Then we observe which ones are type A or A_+. We begin to briefly review of filiform Lie algebras in the following.

Definition 1. *Let* $\mathfrak{g}$ *be an* n*-dimensional Lie algebra. We define* $\mathfrak{g}^1 = \mathfrak{g}, \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}], \dots, \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}], 1 \leq i \leq k$. *The sequence defined in the following form:*

$$\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \mathfrak{g}^2 \supset \mathfrak{g}^3 \supset \cdots \supset \mathfrak{g}^{k-1} \supset \mathfrak{g}^k = \{0\} \tag{1}$$

is called the descending central sequence. If $\mathfrak{g}^k = \{0\}$ *but* $\mathfrak{g}^{k-1} \neq \{0\}$*, then* k *is called nilindex for* $\mathfrak{g}$*.*

Definition 2. *A Lie algebra* $\mathfrak{g}$ *is said to be nilpotent if there exists a positive integer* k *such that* $\mathfrak{g}^{k+1} = \{0\}$*. If* $\mathfrak{g}^k \neq \{0\}$ *for positive integer minimal* k*, then* $\mathfrak{g}$ *is said to be* k*-step nilpotent.*

Definition 3. *An* n*-dimensional nilpotent Lie algebra* $\mathfrak{g}$ *is said to be filiform if its nilindex* k *equal to* $n-1$*.*

Example 4. Three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3 = \langle x, y, z \rangle, [x, y] = z$,is 2-step nilpotent Lie algebra. Since its nilindex $p = 2$ equals $n - 1 = 2$ then $\mathfrak{h}_3$ is filiform Lie algebra as well.

Moreover, if $\mathfrak{g}$ is n-dimensional filiform Lie algebra, then dimension of $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$, equals $n - i$ where $2 \leq i \leq n$. Generally, we have two classes of filiform Lie algebra. Firstly, n-

dimensional standard filiform Lie algebras and secondly, n-dimensional non-standard filiform Lie algebras. Let $S = \{x_1, x_2, x_3, \dots, x_{n-1}, x_n\}$ be a basis for a filiform Lie algebra. The Lie brackets of n-dimensional real standard filiform Lie algebras are given in the following:

$$[x_1, x_k] = x_{k+1}, k = 2,3, .., (n-1). \quad (2)$$

In addition, let $n = 2l + 1$ be odd integer and let $R = \{x_1, x_2, x_3, \dots, x_n, x_{n+1}\}$ be a basis for $(n+1)$ dimensional non-standard real filiform Lie algebra. Their Lie brackets are given as follows:

$$[x_1, x_k] = x_{k+1}, k = 2,3, .., (n-1).$$
$$[x_k, x_{n-k}] = (-1)^k x_n, k = 2, \dots, l. \quad (3)$$

Example 5. The Lie brackets of five-dimensional standard filiform Lie algebra with basis $S = \{x_1, x_2, \dots, x_5\}$ are given by $[x_1, x_2] = x_3, [x_1, x_3] = x_4$, and $[x_1, x_4] = x_5$, while the Lie brackets of five-dimensional non-standard filiform Lie algebra with basis $S = \{x_1, x_2, \dots, x_5\}$ are given by $[x_1, x_2] = x_3, [x_1, x_3] = x_4$, $[x_1, x_4] = x_5$, and $[x_2, x_3] = x_5$.

Definition 6. *Let* $\mathfrak{g}$ *be a Lie algebra. A center of* $\mathfrak{g}$ *is defined as*

$$\mathfrak{z}_1(\mathfrak{g}) = \{x \in \mathfrak{g} \ ; [x, \mathfrak{g}] = 0\}. \quad (4)$$

Furthermore, a second center of $\mathfrak{g}$ *is given in the following form:*

$$\mathfrak{z}_2(\mathfrak{g}) = \{x \in \mathfrak{g} \ ; [[x, \mathfrak{g}], \mathfrak{g}] = 0\}. \quad (5)$$

Definition 7. *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\mathfrak{h}$ *be a subset of* $\mathfrak{g}$. *The centralizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *is given by*

$$\mathfrak{z}(\mathfrak{h}: \mathfrak{g}) = \{x \in \mathfrak{g} \ ; \ [x, \mathfrak{h}] = 0\} \quad (6)$$

Definition 7. *Let* $\mathfrak{n}$ *be a nilpotent Lie algebra with* $\mathfrak{z}(\mathfrak{n}) = 1$. *The Lie algebra* $\mathfrak{n}$ *is called a type A if either the following condition is satisfied:* $[\mathfrak{z}_2(\mathfrak{n}), \mathfrak{z}_2(\mathfrak{n})] = 0$ *or* $\mathfrak{z}_2(\mathfrak{n}) = \mathfrak{n}$.

Example 8 (Beltiţă & Beltiţă, 2015). Let $\mathfrak{g}$ be the four-dimensional standard filiform Lie algebra with basis $S = \{x_1, \dots, x_4\}$. The Lie brackets of $\mathfrak{g}$ are given by

$[x_1, x_2] = x_3, [x_1, x_3] = x_4$.

Then we have that $\mathfrak{g}$ is type A.

Definition 9. *Let* $\mathfrak{n}$ *be a nilpotent Lie algebra with* $\mathfrak{z}(\mathfrak{n}) = 1$. *The Lie algebra* $\mathfrak{n}$ *is called a type* A_+ *if either the following condition is satisfied:* $\mathfrak{z}_2(\mathfrak{n})$ *is a maximal abelian subalgebra of* $\mathfrak{n}$ *or* $\mathfrak{z}_2(\mathfrak{n}) = \mathfrak{n}$.

Remark 10. The subalgebra $\mathfrak{z}_2(\mathfrak{n})$ of $\mathfrak{n}$ is said to be a maximal abelian subalgebra of $\mathfrak{n}$ if $\mathfrak{z}_2(\mathfrak{n}) = \mathfrak{z}_1(\mathfrak{z}_2(\mathfrak{n}):\mathfrak{n})$.

Example 11 (Beltiţă & Beltiţă, 2015). The Lie algebra $\mathfrak{h} = \langle x_1, \dots, x_6 \rangle$ with Lie brackets are $[x_6, x_5] = x_4, [x_3, x_2] = x_1, [x_6, x_4] = x_1$ is of type A_+.

Results and Discussions

The main result in this paper is written in the following proposition:

Proposition 12. *Let $\mathfrak{g}_1$ be a filiform Lie algebra of dimension 3 then $\mathfrak{g}_1$ is of type A and A_+ as well. In general, a standard filiform Lie algebra $\mathfrak{g}_n$ of dimension $n \geq 4$ is neither of type A nor of type A_+.*

Proposition 13. *Let $\mathfrak{g}'$ be the non-standard filiform Lie algebra of dimension 5, then $\mathfrak{g}'$ is of type A but it is not type A_+.*

Before we complete the proof of main results, we complete some properties relate to Definitions 6-7 as follows:

Proposition 14. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{z}_1(\mathfrak{g})$ is an abelian Lie algebra and $\mathfrak{z}_2(\mathfrak{g})$ is 2-step nilpotent Lie algebra which are characteristics ideals of $\mathfrak{g}$. Moreover, we have that $\mathfrak{z}_1(\mathfrak{g})$ is contained in $\mathfrak{z}_2(\mathfrak{g})$. In other words, $\mathfrak{z}_1(\mathfrak{g}) \subseteq \mathfrak{z}_2(\mathfrak{g})$.*

Proof. First we shall prove that $\mathfrak{z}_1(\mathfrak{z}_1(\mathfrak{g})) = \mathfrak{z}_1(\mathfrak{g})$. Let $x \in \mathfrak{z}_1(\mathfrak{z}_1(\mathfrak{g}))$ then $x \in \mathfrak{z}_1(\mathfrak{g}_1)$. Conversely, let $x \in \mathfrak{z}_1(\mathfrak{g})$ then $x \in \mathfrak{g}_1$ and $[x, \mathfrak{g}_1] = 0$. But $[x, y] = 0$ for all $y \in \mathfrak{g}_1$ and since $\mathfrak{z}_1(\mathfrak{g}) \subseteq \mathfrak{g}_1$ then $[x, \mathfrak{z}_1(\mathfrak{g})] = 0$. Therefore, $x \in \mathfrak{z}_1(\mathfrak{z}_1(\mathfrak{g}))$. Thus, $\mathfrak{z}_1(\mathfrak{g})$ is an abelian Lie algebra. Second, we recall the proof that $\mathfrak{z}_2(\mathfrak{g})$ is 2-step nilpotent Lie algebra in (Beltiţă & Beltiţă, 2015). By definition 6 in equation (5) then we have

$\mathfrak{z}_2(\mathfrak{g}) = \{x \in \mathfrak{g} \ ; [[x, \mathfrak{g}], \mathfrak{g}] = 0 \}$.

We shall prove that $\mathfrak{z}_2(\mathfrak{g})^3 = \{0\}$. Since $\mathfrak{z}_2(\mathfrak{g})^2 = [\mathfrak{z}_2(\mathfrak{g}), \mathfrak{z}_2(\mathfrak{g})]$ and $\mathfrak{z}_2(\mathfrak{g}) \subseteq \mathfrak{g}$ then

$$\begin{aligned}\mathfrak{z}_2(\mathfrak{g})^3 &= [\mathfrak{z}_2(\mathfrak{g}), \mathfrak{z}_2(\mathfrak{g})^2] \\ &= [\mathfrak{z}_2(\mathfrak{g}), [\mathfrak{z}_2(\mathfrak{g}), \mathfrak{z}_2(\mathfrak{g})]] \subseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{z}_2(\mathfrak{g})]] = \{0\}.\end{aligned}$$

Thus, $\mathfrak{z}_2(\mathfrak{g})$ is a 2-step nilpotent Lie algebra.

Let der $\mathfrak{g}$ be a set of all Lie algebra derivations of $\mathfrak{g}$. Let $D \in$ der $\mathfrak{g}$. We shall prove that $\mathfrak{z}_1(\mathfrak{g})$ is invariant under D. Let $x, y \in \mathfrak{z}_1(\mathfrak{g})$. Then

$D([x, y]) = [D(x), y] + [x, D(y)] = 0 + 0 = 0 \in \mathfrak{z}_1(\mathfrak{g})$.

Thus, $\mathfrak{z}_1(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$. Similarly, $\mathfrak{z}_2(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$ as well.

Third, let $x \in \mathfrak{z}_1(\mathfrak{g})$ then $[x, \mathfrak{g}] = 0$. But $0 = [x, 0] = [x, [x, \mathfrak{g}]]$, this implies $x \in \mathfrak{z}_2(\mathfrak{g})$. Thus, $\mathfrak{z}_1(\mathfrak{g}) \subseteq \mathfrak{z}_2(\mathfrak{g})$.

We recall result in (Beltiţă & Beltiţă, 2015), particularly in example 6.4 p. 99 which is re-written in Example 11 above. The result is different from result in (Beltiţă & Beltiţă, 2015). The Lie algebra $\mathfrak{h} = \langle x_1, \dots, x_6 \rangle$ with Lie brackets are

$$[x_6, x_5] = x_4, [x_3, x_2] = x_1, [x_6, x_4] = x_1$$

is neither A nor A_+. The Lie algebra $\mathfrak{h}$ is three-step nilpotent Lie algebra. The center of $\mathfrak{h}$ is given by $\mathfrak{z}_1(\mathfrak{h}) = \langle x_1 \rangle$. Therefore, the first center of $\mathfrak{h}$ is one-dimensional. Furthermore, we have $\mathfrak{z}_2(\mathfrak{h}) = \langle x_1, x_2, x_3, x_4, x_6 \rangle \neq \mathfrak{h}$. Indeed, $\mathfrak{z}_2(\mathfrak{h})$ fails to be abelian. Thus, $\mathfrak{h}$ is not of type A. On the other hand, we also have $\mathfrak{z}_1(\mathfrak{z}_2(\mathfrak{h}) : \mathfrak{h}) = \mathfrak{z}_1(\mathfrak{h}) = \langle x_1 \rangle \neq \mathfrak{z}_2(\mathfrak{h})$. Thus, $\mathfrak{h}$ is not of type A_+(see Theorem 4.1 in (Beltiţă & Beltiţă, 2015)).

Proof of Proposition 12. The 3-dimensional filiform Lie algebra is of the form $\mathfrak{g}_1 = \langle x_1, x_2, x_3 \rangle$ with its bracket is $[x_1, x_2] = x_3$. By computing directly, then we have $\mathfrak{g}_1^2 = \langle x_3 \rangle$ and $\mathfrak{g}_1^3 = \{0\}$. Therefore, $\mathfrak{g}_1$ is a 2-step nilpotent Lie algebra or a nilpotent Lie algebra of nilpotence degree equals 2. In addition, $\mathfrak{z}_1(\mathfrak{g}_1) = \langle x_3 \rangle$ which is 1-dimensional first center of $\mathfrak{g}_1$. There exists an ideal $\mathfrak{h}_0 = \langle x_2, x_3 \rangle$, $x_1 \in \mathfrak{g}_1$, $x_2 \in \mathfrak{h}_0$ with $[x_1, x_2] = x_3$ s.t. $\mathfrak{h} = \mathfrak{h}_0 \rtimes \mathbb{R}x_1$. By direct computations, $\mathfrak{z}_2(\mathfrak{g}_1) = \langle x_1, x_2, x_3 \rangle = \mathfrak{g}_1$. Therefore, the Heisenberg Lie algebra $\mathfrak{g}_1$ is of type A. Futhermore, eventhough $\mathfrak{z}_1(\mathfrak{z}_2\mathfrak{g}_1 : \mathfrak{g}_1) = \mathfrak{z}_1(\mathfrak{g}_1) = \langle x_3 \rangle \neq \mathfrak{z}_2(\mathfrak{g}_1)$ but $\mathfrak{z}_2(\mathfrak{g}_1) = \mathfrak{g}_1$. Thus $\mathfrak{g}_1$ is of type A_+.

The n-dimensional ($n \geq 4$) standard filiform Lie algebra is of the form $\mathfrak{g}_n = \langle x_i \rangle_{i=1}^n$ whose bracket $[x_1, x_i] = x_{i+1}$, $i = 2,3,4, \ldots, (n-1)$. The Lie algebra $\mathfrak{g}_n$ is nilpotent Lie algebra of nilpotence degree equals $(n-1)$ whose 1-dimensional center $\mathfrak{z}_1(\mathfrak{g}_n) = \langle x_n \rangle$. Using Kirillov Lemma, there exists an ideal $\mathfrak{g}_n^0 = \langle x_i \rangle_{i=2}^n$, $x_1 \in \mathfrak{g}_n$, $x_{n-1} \in \mathfrak{g}_n^0$ with $[x_1, x_{n-1}] = x_n$ s.t. $\mathfrak{g}_n = \mathfrak{g}_n^0 \rtimes \mathbb{R}x_1$. We observe that the second center of $\mathfrak{g}_n$ is $\mathfrak{z}_2(\mathfrak{g}_n) = \langle x_1, x_{n-1}, x_n \rangle$. Indeed, $\mathfrak{z}_2(\mathfrak{g}_n) \neq \mathfrak{g}_n$ and $[\mathfrak{z}_2(\mathfrak{g}_n), \mathfrak{z}_2(\mathfrak{g}_n)] \neq 0$. Thus, $\mathfrak{g}_n$ is not of type A. It can be shown that $\mathfrak{z}_1(\mathfrak{z}_2(\mathfrak{g}_n) : \mathfrak{g}_n) = \langle x_n \rangle = \mathfrak{z}_1(\mathfrak{g}_n)$ which not equal to $\mathfrak{z}_2(\mathfrak{g}_n) = \langle x_1, x_{n-1}, x_n \rangle$. In the other words, $\mathfrak{z}_2(\mathfrak{g}_n)$ is not maximal abelian Lie algebra. Therefore, $\mathfrak{g}_n$ is not of type A_+.

Example 14. Let $\mathfrak{g}_4$ be four-dimensional standard filiform Lie algebra with basis $S = \{x_1, x_2, x_3, x_4\}$. Their Lie brackets are given by

$$[x_1, x_2] = x_3, [x_1, x_3] = x_4.$$

The Lie algebra $\mathfrak{g}_4$ is a nilpotent Lie algebra of nilpotence degree equals 3 or a 3-step nilpotent Lie algebra. The first center of $\mathfrak{g}_4$ is $\mathfrak{z}_1(\mathfrak{g}_4) = \langle x_4 \rangle$. Moreover, the second center of $\mathfrak{g}_4$ is $\mathfrak{z}_2(\mathfrak{g}_4) = \langle x_1, x_3, x_4 \rangle \neq \mathfrak{g}_4$ which is not abelian. Thus, $\mathfrak{g}_4$ is not of type A. Next, we compute the centralizer of $\mathfrak{z}_2(\mathfrak{g}_4)$ in $\mathfrak{g}_4$, we have $\mathfrak{z}_1(\mathfrak{z}_2(\mathfrak{g}_4) : \mathfrak{g}_4) = \langle x_4 \rangle \neq \mathfrak{z}_2(\mathfrak{g}_4)$. Thus, $\mathfrak{g}_4$ is not of type A_+.

Example 15. Let $\mathfrak{g}_5$ be five-dimensional standard filiform Lie algebra with basis $S = \{x_1, x_2, x_3, x_4, x_5\}$. Their Lie brackets are given by

$$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$$

The Lie algebra $\mathfrak{g}_5$ is a nilpotent Lie algebra of nilpotence degree equals 4 or a 4-step nilpotent Lie algebra. Its semi-direct sum can be written as $\langle x_2, x_3, x_4, x_5 \rangle \rtimes \mathbb{R}x_1$. The first center of $\mathfrak{g}_5$ is $\mathfrak{z}_1(\mathfrak{g}_5) = \langle x_5 \rangle$. Moreover, the second center of $\mathfrak{g}_4$ is $\mathfrak{z}_2(\mathfrak{g}_5) = \langle x_1, x_4, x_5 \rangle \neq \mathfrak{g}_5$. Furthermore, $\mathfrak{z}_2(\mathfrak{g}_5)$ is not abelian. Thus, $\mathfrak{g}_5$ is not of type A. Next, we compute the centralizer of $\mathfrak{z}_2(\mathfrak{g}_5)$ in $\mathfrak{g}_5$, we have $\mathfrak{z}_1(\mathfrak{z}_2(\mathfrak{g}_5) : \mathfrak{g}_5) = \langle x_5 \rangle \neq \mathfrak{z}_2(\mathfrak{g}_5)$. Thus, $\mathfrak{g}_5$ is not of type A_+.

Proof of Proposition 13. The 5-dimensional non-standard filiform Lie algebra is of the form $\mathfrak{g}' = \langle x_i \rangle_{i=1}^5$ with Lie brackets are $[x_1, x_i] = x_{i+1}$, $i = 2,3,4$ and $[x_2, x_3] = x_5$. This is 4-step nilpotent Lie algebra. The first center of $\mathfrak{g}'$ is given by $\mathfrak{z}_1(\mathfrak{g}') = \langle x_5 \rangle$ which is 1-dimensional. Its semi-direct sum can be written as $\langle x_2, x_3, x_4, x_5 \rangle \rtimes \langle x_1, x_2 \rangle$. Direct computations, we have : $\mathfrak{z}_2(\mathfrak{g}') = \langle x_4, x_5 \rangle$ and $\mathfrak{z}_1(\mathfrak{z}_2\mathfrak{g}' : \mathfrak{g}') = \langle x_2, x_3, x_4, x_5 \rangle$. Since $[\mathfrak{z}_2(\mathfrak{g}'), \mathfrak{z}_2(\mathfrak{g}')] = 0$ then $\mathfrak{g}'$ is of type A. Furthermore, $\mathfrak{z}_2(\mathfrak{g}')$ is not maximal abelian Lie algebra since $\mathfrak{z}_2(\mathfrak{g}') \neq \mathfrak{z}_1(\mathfrak{z}_2\mathfrak{g}' : \mathfrak{g}')$. Thus, $\mathfrak{g}'$ is not of type A_+.

A filiform Lie algebra is one of Lie algebra classes and many researchers attracted to study this Lie algebra. For future research, one can develop to investigate type A and type A_+ of non-standard filiform Lie algebras in higher dimensional. In another research, we found that Lie brackets of filiform Lie algebras can be written in other forms. It is different from Lie brackets expressed in this paper. In addition, some researcher develop discrete mathematics in studying of classification of filiform Lie algebras in general over a finite field $\mathbb{Z}/n\mathbb{Z}$, with $n = 2,3,5$. Indeed, this gave a recent research in studying of filiform Lie algebras.

Conclusion

We proved that the standard filiform Lie algebra $\mathfrak{g}_n$ for $n = 3$ known as three-dimensional Heisenberg Lie algebra is of type A and type A_+ as well, while $\mathfrak{g}_n$ for $n \geq 4$ is neither of type A nor A_+. In addition, the five-dimensional non-standard filiform Lie algebra $\mathfrak{g}'$ is of type A but it is not type A_+.

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Conflicts of interest
The authors declare that there are no conflicts of interest.

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