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Efficiency of new Canonical polynomials in Solving nonlinear Fractional Integro-Differential equations

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ABSTRACT

This paper is aimed to solve nonlinear fractional integro-differential equations, specifically of the Volterra-types, utilizing newly constructed versatile canonical polynomials. The technique involves the use of the Lanczos method. The popular numerical method known as the collocation method is presented to evaluate the evolving equations and subsequently to determine the values of the embedded unknown coefficients. The equations exhibit both derivatives and integrals. The resulting approximate solutions are compared with the given exact solutions. Numerical experiments are conducted to showcase the efficiency and accuracy of the technique, which is achieved by estimating the errors in the approximate solutions in order to significantly establish the convergence of the method. The mathematical tool utilized to obtain the required results is Maple 18 software package.

Keywords: Lanczos Methods; Canonical Polynomials; Riemann-Liouville; Taylor series;
Subject Classification Code: Standard Collocation methods.

Introduction

Series of nonlinear equations had been visited and desirable solutions were achieved analytically. For instance in (Ogunbamike et al., 2019; Shahid & Shah, 2024), whereby the convergence of differential transform method was investigated on nonlinear systems of equations; and operational matrix of integration based on the Taylor wavelet technique was adopted to solve nonlinear Stratonovich-Volterra integral equations respectively. The solution of nonlinear integral equations was also verified by (Pooja et al., 2025) whereby the approximation of unknown functions was carried out using the generalized Bell polynomials. The reliability and the performance of the methods were showcased by the results. However, where analytical methods are incapacitated to handle situations related to these, the problems would not be absolutely solved, hence there is need to consider the behaviours of the equations close to the initial points as good approximations. In recent years, many situations pertinent to this, particularly in Physics and Engineering fields such as fluid mechanics, solid dynamics, solid state physics e.t.c., that lead to integral equations were not easily analytically solved as investigated in (Li & Huang, 2016). Therefore with the aid of numerical methods, qualitative approaches have been explicitly considered using a variety of methods such as the Collocation and Block methods by different authors to overcome the barriers. This is confirmed in (Owolanke et al., 2017; 2019; 2021; Yakusak & Owolanke, 2017; Okedayo et al., 2018) to be mentioned but few; they are all based on the Collocation and Block numerical techniques.

Integro-differential equations are described as integral equations in terms of derivatives. Its special kind is the Volterra-type, which appears when initial value problems are converted to integral equations, such that at least one of the limits of the integration is a variable. Many scientific and engineering applications are described using the equation, which has been verified in population growth models, diffraction models, heat radiation, water waves, scattering in quantum mechanics, and electromagnetic scattering problems. Researchers have worked towards investigating reliable solutions to the problems posed in terms Volterra integro-differential equations in the years past and recently; to mention but few are (Shayamford et al., 2020; Suayip & Gamze, 2022; Aourir & Dastjerdi, 2024; Amam et al., 2024). The numerical technique adopted by the aformentioned authors is the popular Collocation method.

Furthermore, one of the conventional approaches adopted for solving nonlinear problems is the method of linearization. The technique is very effective, particularly while analyzing the stability of fixed points of nonlinear problems subject to Taylor series expansion. The technique is adopted to solve a system of nonlinear equations that emerged from a Biological population model in (Morgan, 2015); the method is also used in (Lei, 2020), whereby a sequence of nonlinear equations is transformed into a system of linear equations using Newton's method. Due to its resulting cumbersomeness in terms of computational costs and inconsistencies, scholars have however diversified, developing a series of numerical methods, providing approximate solutions to the embattled barrier. For instance, Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) are versatile techniques for solving nonlinear fractional differential equations; they are highly rated and consequently considered to be relevant due to their efficiencies and effectiveness in handling the differential equations, by supplying approximations that rapidly converge. A notable example is the Elzaki-Adomian Decomposition Method (EADM), which is a hybrid of the Elzaki transform and Adomian decomposition method. It solves nonlinear partial differential equations as revealed in (Orapine et al., 2022).

In addition, another useful method that is applicable in finding an approximate solution to nonlinear differential equations is the Variational Iteration Method (VIM) as verified by a few authors; among them is (Bonnano et al., 2021), whereby a sixth order nonlinear ordinary differential equations is investigated to verify the existence of nontrivial solutions. Another popular method, specifically for this purpose is the Collocation method; whose technique involves defining equations in terms of coefficients on a set of points within an interval of consideration in order to generate systems of equations. The equations are subsequently solved finding the values of the coefficients in any given assumed solutions. For instance, a Spectral Collocation Method (SCM) is applied (Zhou & Dai, 2021) to analyze a coupled system of nonlinear fractional differential equations, where the given system is first reduced to a system of integral equations before it is discretized; Legendre polynomials are used as the basis function for the approximate solution. The collocation method is also used in (Lei et al., 2020) to solve nonlinear fractional delay differential equations based on Legendre multiwavelets; the error estimation of the approximate solutions is shown. Moreover, the wavelet collocation method was used in (Shahid et al., 2024) to solve perturbed difference equations.

Nonlinear fractional integro-differential equations are mathematical models often encountered in various fields of Science and Engineering. One of the numerical techniques used is the collocation method, as discussed in (Parisa & Yadollah, 2020; Bragdi, 2020; Khalid et al., 2020; Pooja et al., 2023; Shah et al., 2023; Pooja & Shah, 2023). Canonical polynomials illustrated in (Owolanke et al., 2019) are constructed to provide approximate solutions to some linear fractional integro-differential equations. The polynomial was introduced by Lanczos alongside Tau method; the recursive formulation was carried out in (Ortiz, 1969). The method is however extensively discussed in (Lanczos,1956; 1973), whereby the solution to a first order differential equation is investigated by approximating the derivatives and the unknown function by the canonical polynomial to a certain degree. Its effectiveness is confirmed when the interval of integration is subdivided into definite sub-intervals while the approximate value of the unknown function is denoted by $y_p(x)$ in x_p. Therefore, with the given papers mentioned above, this paper is focused on the numerical solution of nonlinear fractional integro-differential equations of the Volterra-type, using new canonical polynomials. The Volterra integro-differential equations has been investigated in (Taiwo et al., 2023), using the Galerkin Method with orthogonal polynomials as basis functions, however the newly constructed Canonical polynomials are utilized in this paper as the basis function so as to establish the convergence of nonlinear equations.

Description of Fractional Derivatives

The Riemann-Liouville and Caputo methods are very fundamental in any study of the theory of fractional derivatives. The Riemann-Liouville fractional derivative denoted by $w^n f(t)$ is defined as follows

$$w^n f(t) = w^n Q^q, \; q > 0 \tag{1}$$

Equivalently written as

$$w^n f(t) = w^n \frac{1}{\Gamma(n-q)} \int_0^x (t-\tau)^{n-q-1} f(\tau) d\tau, \;\; n-1 < q < n \tag{2}$$

Hence, the following properties hold as follow:

1. $w^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta-\alpha}$ (3)
2. $j^\alpha \omega^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!}, \; x > 0, \; m-1 < \alpha < m$ (4)

Methods

Considering nth-order fractional integro-differential equations of the form

$$K^\alpha y(x) + A(x)y(x) + B(x)y^{'}(x) + C(x)y^{''}(x) + \cdots + N(x)y^n(x) + \int_0^x \frac{y(s)}{(k-x)} ds = f(x), \tag{5}$$

where,

$$K^\alpha y(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x (x-q)^{m-q-1} D^m f(\tau) d\tau, & m-1 < q < m \\ \dfrac{d^m f(x)}{dx^m}, \;\; q = m, m \in \aleph \end{cases} \tag{6}$$

equation (5) becomes

$$K^\alpha y(x) + A(x)y(x) + B(x)y^{'(x)} + C(x)y^{''}(x) + \cdots + N(x)y^n(x) = F(x) \tag{7}$$

Defining the differential operator D as

$$A\frac{d^\alpha}{dx^\alpha} + B\frac{d^2}{dx^2} + C\frac{d^3}{dx^3} + \cdots + N\frac{d^n}{dx^n} \equiv D \tag{8}$$

$$Dx^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} x^{m-\alpha} + Ax^m + B(m)x^{m-1} + C(m)(m-1)x^{m-2} + \cdots + N(m)(m-1)(m-2)\ldots(m-k+1)x^{m-k} \tag{9}$$

by Lanczos (1956)

$$Dp_m(x) = x^m, \qquad m = 0,1,2, \tag{10}$$

Then, equation (9) becomes

$$Dx^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} Dp_{m-\alpha}(x) + ADp_m(x) + B(m)Dp_{m-1}(x) + C(m)(m-1)Dp_{m-2}(x) + \cdots + N(m)(m-1)(m-2)\ldots(m-k+1)Dp_{m-k}(x) \tag{11}$$

Assuming the inverse of the operator in equation (11) exists, the equation becomes

$$x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} p_{m-\alpha}(x) + Ap_m(x) + B(m)p_{m-1}(x) + C(m)(m-1)p_{m-2}(x) + \cdots + N(m)(m-1)(m-2)\dots(m-k+1)p_{m-k}(x) \quad (12)$$

Equation (12) can equivalently be written as

$$p_m(x) = \frac{1}{A}\{\ x^m - \left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} p_{m-\alpha}(x) + B(m)p_{m-1}(x) + C(m)(m-1)p_{m-2}(x) + \cdots + N(m)(m-1)(m-2)\dots(m-k+1)p_{m-k}(x)\right]\} \quad (13)$$

Further simplification implies that

$$p_m(x) = \frac{1}{A}\{\ x^m - \left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} p_{m-\alpha}(x) + B(m)p_{m-1}(x)\right]\} \quad (14)$$

is the required canonical polynomial of first order fractional integro-differential equations while that of the second order is truncated after the term
$C(m)(m-1)p_{m-2}(x)$.

It implies that the Canonical polynomial for the second order fractional integro-differential equations as deduced from equation (13) is

$$p_m(x) = \frac{1}{A}\{\ x^m - \left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} p_{m-\alpha}(x) + B(m)p_{m-1}(x) + C(m)(m-1)p_{m-2}(x)\right]\} \quad (15)$$

Standard Collocation Method

This is a popular numerical method of solution of differential and integral equations. The technique is implemented in this paper to determine the values of the unknown constants in the assumed solution. Let the approximate solution for equation (5) be given as

$$g(x) \approx g_N(x) = \sum_{j=0}^{N} c_j z_j(x), \quad j \geq 0 \quad (16)$$

where the coefficients to be determined are $c_j's$, and $z_j^{'}s$ represent the constructed canonical polynomials in equation (13). The degree of the approximation is denoted as N. Hence, substituting equation (16) into equation (5) yields

$$k^\alpha \sum_{j=0}^{N} c_j z_j(x) + A(x)\sum_{j=0}^{N} c_j z_j(x) + B(x)\sum_{j=0}^{N} c_j z_j{}'(x) + C(x)\sum_{j=0}^{N} c_j z_j^{''(x)} + \cdots + N(x)\sum_{j=0}^{N} c_j z_j{}^n(x) + \lambda\int_0^x \frac{g(s)}{(k-x)}\,ds = f(x) \quad (17)$$

where, $g(s)$ is determined by the Taylor series approach in order to evaluate the integral.

Let $g(s) = g_N(x) + (s-x)g_N^{'}(x) + \frac{(s-x)^2}{2!}g_N^{''}(x) + \cdots + \frac{(s-x)^n}{n!}g_N{}^n(x)$ (18)

Thus, the integral part of equation (17) becomes

$$\int_0^x \frac{1}{(k-x)}\left[g_N(x)+(s-x)g_N'(x)+\frac{(s-x)^2}{2!}g_N''(x)+\cdots+\frac{(s-x)^n}{n!}g_N{}^n(x)\right]ds \tag{19}$$

where,

$$g_N(x)=c_0z_0(x)+c_1z_1(x)+c_2z_2(x)+\cdots+c_Nz_N(x)$$
$$g_N{}'(x)=c_0z_0{}'(x)+c_1z_1{}'(x)+c_2z_2{}'(x)+\cdots+c_Nz_N{}'(x)$$
$$g_N{}''(x)=c_0z_0{}''(x)+c_1z_1{}''(x)+c_2z_2{}''(x)+\cdots+c_Nz_N{}''(x)$$

Hence, the nth-derivative is written as

$$g_N{}^n(x)=c_0z_0{}^n(x)+c_1z_1{}^n(x)+c_2z_2{}^n(x)+\cdots+c_Nz_N{}^n(x) \tag{20}$$

Solving equation (17) in terms of equations (18) and (19), it gives

$$K^\alpha[c_0z_0(x)+c_1z_1(x)+c_2z_2(x)+\cdots+c_Nz_N(x)]+A(x)[c_0z_0(x)+c_1z_1(x)+c_2z_2(x)+\cdots+c_Nz_N(x)]+B(x)[c_0z_0{}'(x)+c_1z_1{}'(x)+c_2z_2{}'(x)+\cdots+c_Nz_N{}'(x)]+C(x)\left[c_0z_0^{''(x)}+c_1z_1^{''(x)}+c_2z_2^{''(x)}+\cdots+c_Nz_N^{''(x)}\right]+\cdots+N(x)[c_0z_0{}^n(x)+c_1z_1{}^n(x)+c_2z_2{}^n(x)+\cdots+c_Nz_N{}^n(x)]+G(x)=f(x) \tag{21}$$

where $G(x)$denotes equations (18), (19), and system of equation (20).

Collocating equation (21) at $x=x_k$, such that $x_k=x_0+\frac{(x_n-x_0)k}{j+1}$, k=1,2,...,j to give

$$K^\alpha[c_0z_0(x_k)+c_1z_1(x_k)+c_2z_2(x_k)+\cdots+c_Nz_N(x_k)]+A(x_k)[c_0z_0(x_k)+c_1z_1(x_k)+c_2z_2(x_k)+\cdots+c_Nz_N(x_k)]+B(x_k)[c_0z_0{}'(x_k)+c_1z_1{}'(x_k)+c_2z_2{}'(x_k)+\cdots+c_Nz_N{}'(x_k)]+C(x_k)[c_0z_0{}''(x_k)+c_1z_1{}''(x_k)+c_2z_2{}''(x_k)+\cdots+c_Nz_N{}''(x_k)]+\cdots+N(x_k)[c_0z_0{}^n(x_k)+c_1z_1{}^n(x_k)+c_2z_2{}^n(x_k)+\cdots+c_Nz_N{}^n(x_k)]+G(x_k)=f(x_k) \tag{22}$$

Numerical Experiments

In this section, a few examples are considered in order to confirm the effectiveness of the method.

Example 1: $D^{\frac{3}{2}}y(x)+2\left(\frac{dy}{dx}\right)^2+x^2\frac{d^2y}{dx^2}+\int_0^x\frac{y(t)}{(x-t)^{\frac{1}{3}}}\,dt=4+1.128379167x^{0.5}+\frac{3}{2}x^{\frac{2}{3}}+8x+0.7522527778x^{1.5}+\frac{9}{10}x^{\frac{5}{3}}+7x^2+\frac{10}{3}x^3+\frac{81}{880}x^{\frac{11}{3}}+\frac{27}{80}x^{\frac{8}{3}}\quad 0\le x\le 1\quad y(0)=y'(0)=0$

Exact solution: $y(x)=e^x$

Example 2:

$$D^{\frac{1}{2}}y(x)+\frac{dy}{dx}+\int_0^x 2y^2(t)dt$$
$$=10+11.28379187x^{0.5}-500x^2-300.9011112x^{2.5}+66.66666667x^3+4166.666666x^4+1910.483246x^{4.5}-1333.333334x^5-13888.88889x^6-5344.009079x^{6.5}+12698.41270x^7-70546.737222x^9+200000.465127465x^{11}-500000.0875505086x^{13}+-500000.249013185x^{15}\quad 0\le x\le 1\quad y(0)=y'(0)=0$$

Exact solution: $y(x)=\sin(10x)$

Example 3:

$\frac{d^2y(x)}{dx^2}+\frac{d^{1.5}y(x)}{dx^{1.5}}+y^2(x)+\int_0^x\frac{y(t)}{\sqrt{x-t}}dt=18cos(3x)^2$- $18sin(3x)^2+20.3108250x^{0.5}-1671.290743x^{4.5}-13899.41906x^{8.5}+79819.57292x^{16..5}+sin(3x)^4+\frac{48}{5}sin(x)^{\frac{5}{2}}$

$$\textbf{\textit{Exact solution}}: \quad y(x) = \sin(3x)^2$$

Discussions of Results

Based on the numerical results obtained from the three tested examples, the constructed canonical polynomial method demonstrates excellent performance in solving nonlinear fractional integro-differential equations of Volterra-type.

Accuracy Analysis

Example 1 exhibits exceptionally high accuracy with a maximum absolute error of 2.8381E-06 at x = 0.7. This remarkably small error indicates that the method provides an approximation very close to the exact solution y(x) = e^x. The relatively stable error pattern throughout the interval [0,1] demonstrates the method's consistency in maintaining accuracy across the computational domain.

Example 2 shows satisfactory performance with the largest absolute error of 1.2248E-02 at the endpoint x = 1.0. Although this error is larger compared to Example 1, it remains within acceptable tolerance limits for numerical solutions. The increasing error trend toward the interval's end suggests that the method maintains good stability in the initial portion of the interval but requires special attention for longer computational domains.

Example 3 presents more significant error variation, with a maximum error of 9.5008E-02 at x = 0.3. Despite this relatively larger error, the convergence pattern observed at the endpoint (zero error at x = 1.0) demonstrates that the method can still handle equations with higher complexity effectively.

Convergence and Stability

The graphs presented in Figures 1 clearly show that the approximate solutions follow the exact solution patterns very closely. This confirms the convergence of the developed canonical polynomial method. Numerical stability is evident from the absence of spurious oscillations or irregular fluctuations in the approximate solutions.

Computational Efficiency

The use of different polynomial degrees (N=11 for Example 1, N=14 for Example 2, and N=13 for Example 3) demonstrates the method's flexibility in adapting to problem complexity (Table 1-3). This approach allows for controlling the balance between accuracy and computational burden.

Table 1. Absolute Errors of Example 1 for Case N=11

x	Exact Solution	Approximate Solution	Absolute Errors
0.000	0.00000000	0.00000000	0.00000000
0.100	1.10517092	1.105170888	$3.2201E-08$
0.200	1.22140276	1.221402667	$9.2954E-08$
0.300	1.34985881	1.349858796	$1.4298E-08$
0.400	1.49182470	1.491824540	$1.5958E-07$
0.500	1.64872127	1.648721136	$1.3365E-07$
0.600	1.82211880	1.822118128	$6.7224E-07$
0.700	2.01375271	2.013749872	$2.8381E-06$
0.800	2.22554093	2.225539592	1.3376E − 06
0.900	2.45960311	2.459600891	2.2190E − 06
1.000	2.71828183	2.718279251	$2.5790E-06$

Advantages of the Developed Method

- High Accuracy: Errors in the order of 10^{-6} to 10^{-2} demonstrate excellent precision for practical applications.
- Versatility: The method successfully handles various types of nonlinear fractional integro-differential equations with different characteristics.

- Numerical Stability: No significant numerical instabilities are observed throughout the computational intervals.
- Direct Implementation: The method does not require prior linearization processes, making it more efficient in application.

Table 2. Absolute Errors of Example 2 for Case N=14

x	Exact Solution	Approximate Solution	Absolute Errors
0.000	0.00000000	0.000005800	$5.8000E-06$
0.100	0.84147098	0.8414691884	$1.7916E-06$
0.200	0.90929743	0.9092946185	$2.8115E-06$
0.300	0.14112001	0.1411187640	$1.2460E-06$
0.400	-0.75680250	-0.7568061952	$3.6952E-06$
0.500	-0.95892427	-0.9589272386	$2.9686E-06$
0.600	-0.27941550	-0.2794165040	$1.0040E-06$
0.700	0.65698660	0.6569839484	$2.6516E-06$
0.800	0.98935825	0.989326216	3.2034E − 05
0.900	0.41211849	0.41080049	1.3180E − 03
1.000	-0.54402111	-0.55626911	$1.2248e-02$

Table 3. Absolute Errors of Example 3 for Case N=13

x	Exact Solution	Approximate Solution	Absolute Errors
0.000	0.00000000	0.00000000	0.00000000
0.100	0.08733219	0.12900742	$4.1675E-02$
0.200	0.31882112	0.40644702	$8.7626E-02$
0.300	0.61360105	0.70860909	$9.5008E-02$
0.400	0.86869686	0.92883996	$6.0143E-02$
0.500	0.99499625	0.99944563	$4.4494E-03$
0.600	0.94837921	0.90579454	$4.2585E-02$
0.700	0.74513041	0.68475017	$6.0380E-02$
0.800	0.45625051	0.40910898	4.7142E − 02
0.900	0.18265356	0.16362805	1.9026E − 02
1.000	0.01991486	0.01991486	$0.0000E+00$

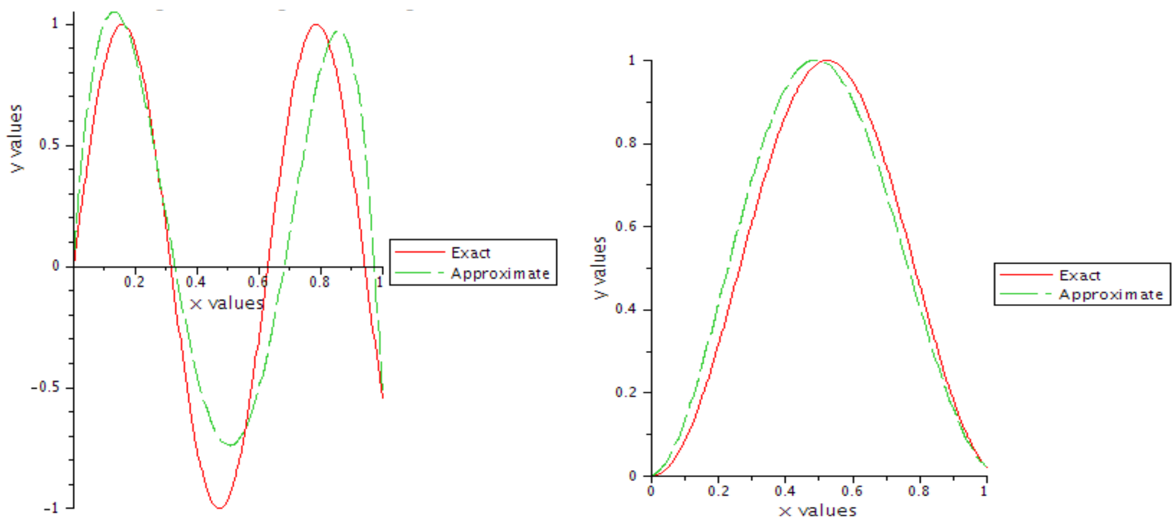

Figure 1. The comparism graph of Example (a) 2 and (b) 3

Comparison with Other Methods

Compared to conventional methods that require linearization, the developed canonical polynomial approach offers advantages in terms of:

- Avoiding additional computational complexity from linearization processes

- Preserving the nonlinear characteristics of the original equations
- Providing more accurate approximations for nonlinear problems

Error Behavior and Convergence Characteristics

The error analysis reveals distinct patterns across the three examples:

- Example 1 shows uniform convergence with consistently small errors, indicating excellent stability for exponential-type solutions.
- Example 2 demonstrates good initial accuracy with gradual error growth, suggesting the need for adaptive strategies for trigonometric functions over extended domains.
- Example 3 exhibits variable error patterns but maintains overall convergence, confirming the method's robustness for complex nonlinear scenarios.

Practical Implications

The results demonstrate that the canonical polynomial method is particularly well-suited for:

- Problems requiring high precision in moderate intervals
- Nonlinear systems where linearization introduces significant approximation errors
- Applications in physics and engineering where fractional derivatives model memory effects

Limitations and Recommendations

While the method shows excellent performance, several aspects warrant attention:

- Polynomial Degree Selection: The optimal polynomial degree should be determined based on problem complexity and desired accuracy levels. Higher degrees may improve accuracy but increase computational cost.
- Computational Domain: For longer intervals, domain subdivision or parameter adjustment may be necessary to maintain accuracy, particularly evident from Example 2's behavior.
- Further Validation: Testing on broader classes of nonlinear fractional integro-differential equations would strengthen the method's validity and establish its general applicability.
- Adaptive Strategies: Implementation of adaptive degree selection or interval subdivision could enhance the method's robustness for challenging problems.

Comparison with Literature

The achieved accuracy levels compare favorably with existing methods reported in the literature. The direct approach to nonlinear problems without linearization represents a significant advancement over traditional techniques, offering both computational efficiency and solution accuracy.

Conclusion

In this paper, the numerical solution of nonlinear fractional integro-differential equations of the Volterra-type was investigated, in which new canonical polynomials were constructed and used as basis functions. The effectiveness of the approach is demonstrated on numerical examples with initial conditions without first undergoing the process linearization. The outcome of the findings is presented by means of the depicted table of results and graphs. In view of all of these, it can be inferred that the approach is reliable to handle equations of this kind.

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Conflicts of interest

We hereby declare that there is no conflict of interest in the course of this research work.

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