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Legendre Collocation approach for Integro-Differential equations

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ABSTRACT
This study presents the application of the Legendre Collocation Method (LCM) for solving Integro-Differential Equations (IDEs), which model a range of scientific and engineering problems. IDEs, involving both differential and integral terms, often require numerical methods for their solutions due to the complexity of obtaining exact solutions. The proposed approach transforms IDEs into systems of linear algebraic equations using shifted Legendre polynomials. By collocating the resulting equations, approximate solutions are efficiently computed. The accuracy of the method is validated through several numerical examples, including Volterra and Fredholm types of IDEs, and the results are compared with known exact solutions. The effectiveness and robustness of LCM are demonstrated through high-order approximations. The theoretical uniqueness of the method is established using relevant theorems, including the Banach Contraction Principle. Overall, the LCM provides a reliable and efficient technique for solving a wide class of IDEs with high accuracy.

Keywords:
polynomials; integro-differential equations; approximate solutions; collocation method

Introduction

Integro-Differential Equations (IDEs) are mathematical expressions that incorporate both integral and differential operations. These equations are essential in various scientific and engineering disciplines because they model systems where current states depend on both rates of change and accumulated histories. By including integral terms, IDEs generalize differential equations, often resulting in more complex dynamics and solutions.

IDEs are applicable in numerous fields, including the description of the motion of viscoelastic materials, population dynamics in biology, heat conduction with memory effects, and financial mathematics. Solving these equations typically requires specialized techniques, including numerical methods, as closed-form solutions are often not feasible. Studying IDEs provides deeper insights into systems that exhibit both instantaneous and cumulative responses.
responses.

Recently, there has been growing attention on the study of IDEs because they are often difficult to handle mathematically, and many of them do not admit closed-form solutions. As a result, developing reliable numerical approximations has become essential. Several researchers have proposed different computational strategies for IDEs, depending on the form of the equation being studied. For example, Ren et al. (1999) examined a group of second-kind integral equations by employing a Taylor-series-based approach. Maleknejad and Hadizadeh (1999) developed a numerical procedure tailored for Volterra-Fredholm IDEs.

Maleknejad and Aghazadeh (2005) presented an approximate technique for second-kind Volterra equations with convolution kernels using a Taylor-series framework. Babolian and Davari (2005) used the Adomian Decomposition Method (ADM) for linear Volterra IDEs of the second kind, while Chakrabrti and Marttha (2009) produced approximate solutions for Fredholm IDEs of the same category. Mostefa and Mustapha (2017) employed Euler-type series for linear IDEs. Mehdiyera and Imanova (2015) approached Volterra IDEs by implementing a second-derivative-based strategy. Deniz and Nurcan (2022) utilized the Lucas Collocation Method for high-order linear Fredholm IDEs, and Samaher (2021) adopted an iterative scheme for Volterra-Fredholm equations.

Uwaheren et al. (2020) proposed a modified collocation technique to handle singular multi-order fractional Lane-Emden equations, while Ayinde et al. (2022) used Chebyshev polynomials of the third kind to address higher-order IDEs. Ajileye et al. (2023) adopted a collocation-driven numerical approach for multi-order fractional IDEs. Oyedepo et al. (2022) applied a least-squares Chebyshev collocation technique for systems of linear fractional IDEs, and Oyedepo et al. (2023) used the homotopy perturbation approach for fractional Volterra and Fredholm models.

Jafarzadeh and Keramati (2018) solved a system of IDEs using a Taylor-collocation technique supported by convergence analysis. Olayiwola et al. (2020) used Legendre polynomials within a collocation framework for Volterra IDEs, while Adesanya et al. (2023) proposed an approximation method for high-order Fredholm IDEs. Issa et al. (2024a) addressed generalized delay IDEs using a Galerkin scheme based on Vieta-Lucas polynomials, and Issa et al. (2024b) developed an analytical approximation for fractional-order generalized IDEs via shifted Vieta-Lucas polynomials. Toma and Postavaru (2023) applied Fibonacci polynomial basis functions to fractional Fredholm-Volterra equations, whereas Ansari and Ahmad (2023) explored nonlinear Volterra-Fredholm IDEs through both the standard and modified ADM.

Although these studies have produced valuable results, many of the existing methods still face challenges related to computational cost and overall accuracy. Therefore, there remains a strong need for more efficient and dependable numerical schemes for IDEs. This study contributes toward filling this need by applying the Legendre Collocation Method (LCM) to Volterra and Fredholm IDEs, using shifted Legendre polynomials to obtain highly accurate approximate solutions.

The type of problem examined in this study can be represented in the form:

$$\nu(\eta) - \gamma \int_0^{\eta} k(\eta,\tau)\upsilon(\tau)d\tau \ = s(\eta)\,, \quad 0 \le \eta \le 1. \tag{1}$$

In Eq. (1) the parameter γ and function $k(\eta,\tau)$ and $s(\eta)$ are given, $\nu(\eta)$ is the to be determined.

Objectives

1. To develop a numerical method using the Legendre Collocation Method (LCM) for solving integro-differential equations (IDEs).
2. To transform IDEs into systems of linear algebraic equations through collocation using shifted Legendre polynomials.
3. To validate the proposed method by applying it to various numerical examples and comparing the results with known solutions.
4. To demonstrate the effectiveness and accuracy of the LCM in solving both Volterra and Fredholm types of IDEs.
5. To establish the theoretical foundation and uniqueness of the method through relevant theorems and hypotheses.
6. To provide a comprehensive comparison of approximate solutions obtained via LCM against exact solutions and other numerical methods to highlight its robustness.

Materials and Methods

Definition 1: Legendre polynomials of degree r are denoted by:

$$\upsilon_r(\eta) = \sum_{i=0}^{[\frac{r}{2}]}(-1)^r \frac{(2r-2i)!}{2^r.i!(r-i)!(r-2r)!}\eta^{r-2i} ,$$

where

$$\left[\frac{r}{2}\right] = \begin{cases} \frac{r}{2} & if \;\; r \; is \; even \\ \frac{r-1}{2}, if \; r \; is \; odd. \end{cases}$$

$r\upsilon_r(\eta) = (2r-1)\eta\upsilon_{r-1}(\eta) - (r-1)\upsilon_{r-2}(\eta); r \geq 2$, starting with

$\upsilon_0(\eta) = 1, \upsilon_1(\eta) = \eta$

Accordingly, the initial Legendre polynomials defined on the interval $[-1,1]$ can be written as:

$$\left.\begin{aligned} \upsilon_0(\eta) &= 1 \\ \upsilon_1(\eta) &= \eta \\ \upsilon_2(\eta) &= \frac{1}{2}(3\eta^2 - 1) \\ \upsilon_3(\eta) &= \frac{1}{2}(5\eta^3 - 3\eta) \end{aligned}\right\} \quad (2)$$

The corresponding shifted Legendre polynomials, adjusted for the interval $[0,1]$, take the form:

$$\left.\begin{aligned} \upsilon^*{}_0(\eta) &= 1 \\ \upsilon^*{}_1(\eta) &= 2\eta - 1 \\ \upsilon^*{}_2(\eta) &= 6\eta^2 - 6\eta + 1 \\ \upsilon^*{}_3(\eta) &= 20\eta^3 - 30\eta^2 + 20\eta - 1 \end{aligned}\right\} \quad (3)$$

Definition 2: Approximate Solution: An approximate solution is an estimation or educated guess of a value, quantity, or solution to a problem that isn't obtained with exact precision but is close enough to the true value to be practically helpful. In fields such as mathematics, science, engineering, and computing, it's common to encounter problems that are too complex, nonlinear, or lack analytical solutions, making exact solutions difficult or impossible. In these situations, an approximate solution provides a practical means to gain insights, make predictions, or solve problems with an acceptable level of accuracy.

Definition 3: An exact solution is a precise and rigorous mathematical expression that fully satisfies a given problem or equation. In mathematical, scientific, and engineering contexts, finding an actual solution is highly valued as it provides a clear and comprehensive description of the problem. An exact solution adheres strictly to the principles and conditions of the problem, leaving no room for uncertainty or approximation.

Definition 4: A matrix on a set W is a function $q \coloneqq : W \times W \rightarrow P$ with the following properties. For all $\alpha, \beta \in W$

(a) $q(\eta, \beta) \geq 0$;
(b) $q(\eta, \beta) = 0 \leftrightarrow \eta = \beta$
(c) $q(\eta, \beta) = q(\beta, \eta)$
(d) $q(\eta, \beta) \leq q(\eta, \vartheta) + q(\eta, \beta)$

If d is a matrix on W, then the pair (W, q) is called the matrix space.

Definition 5:Let (B, q) be a matrix space. A mapping $T: \mathrm{B} \rightarrow \mathrm{B}$ is Lipschitzian if there exists a constant $L > 0$ such that $q\left(B_\eta, B_\beta\right) \leq Lq(\eta, \beta) \forall \eta, \beta \epsilon \mathrm{B}$.

Proposed method

To obtain a numerical approximation for the type of integro-differential problems addressed in this work, we represent the solution as an expansion in shifted Legendre basis functions, expressed as:

$$v(\eta) = \sum_{i=0}^{r} v^*{}_i(\eta)\, a_i, \tag{4}$$

Here, a_i , $i = 0(1)r$, denote the unknown coefficients that must be evaluated. Substituting expression (4) into equation (1) results in

$$\sum_{i=0}^{r} v^*{}_i(\eta)\, a_i - \gamma \int_0^\eta k(\eta, \tau) \sum_{i=0}^{r} v^*{}_i(\eta)\, a_i d\tau \;= s(\eta) \tag{5}$$

Define $\varphi(\eta) = \sum_{i=0}^{r} v^*{}_i(\eta)\, a_i$ and $\varrho(\eta) = \int_0^\eta k(\eta, \tau) v(\tau) d\tau$.

Under these definitions, equation (5) becomes

$$\varphi(\eta) - \varrho(\eta) = s(\eta) \tag{6}$$

A system of linear algebraic equations involving the (r+1) unknown coefficients is generated by enforcing equation (6) at a set of uniformly distributed collocation points

$\eta_i = m + \frac{(m-n)i}{r}$, $(i = 0(1)r)$.

This yields the matrix form

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \cdots & \Psi_{1n} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} & \cdots & \Psi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{m1} & \Psi_{m2} & \Psi_{m3} & \Psi_{m3} & & \Psi_{mn} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \mathrm{E}_{11} \\ \mathrm{E}_{21} \\ \vdots \\ \vdots \\ \mathrm{E}_{mn} \end{pmatrix} \tag{7}$$

where the $\Psi_i{}'s$ represent the entries multiplying the coefficients a_{is}, and the $\mathrm{E}_i{}'s$ correspond to the values of $s(\eta_i)$. .

To determine the coefficients, the resulting system is solved using a matrix-inversion procedure, expressed as

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \cdots & \Psi_{1n} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} & \cdots & \Psi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{m1} & \Psi_{m2} & \Psi_{m3} & \Psi_{m3} & & \Psi_{mn} \end{pmatrix}^{-1} \begin{pmatrix} E_{11} \\ E_{21} \\ \vdots \\ \vdots \\ E_{mn} \end{pmatrix} \quad (8)$$

Finally, the approximate numerical solution is obtained by inserting these computed coefficients back into the assumed representation in equation (4).

Theorem: (The Banach Contraction Principle) Let (B, q) be a complete matrix space, and each contraction mapping $T: B \to B$ has a unique fixed point η of T in B, such that $T\eta = \eta$

Uniqueness of the Method

In this section, we established the uniqueness of the method by introducing the following theorem and hypothesis:

H_1 : There exist a constant, $L > 0$ such that for υ_R and $\upsilon \in C([0,1], \mathcal{R})$

$$|F(\upsilon_R) - F(\upsilon)| \le |\upsilon_R - \upsilon|$$

H_2 : There exist a function $k^* \in C([0,1] \times [0,1], \mathbb{R})$ the set of all possible function such that

$k^* = \max_{\eta \in [0,1]} \int_0^\infty |k(\eta,\tau)|\, d\tau < \infty$

H_3 : The function $g \in \mathbb{R}$ is continuous.

Theorem 5.1 Assume $H_1 - H_3$ hold.

$$\left(\frac{Lk^*}{\Gamma(\alpha+1)}\right) < 1 \quad (9)$$

Then there exists a unique solution $\upsilon(\eta) \epsilon T$

Proof

$$(T\upsilon_R)(\eta) = \int_0^\eta k(\eta,\tau) F(\upsilon_R((\tau)))d\tau \; + s(\eta) \quad (10)$$

$$(T\upsilon_r)(\eta) = \int_0^\eta k(\eta,\tau) F(\upsilon_r((\tau)))d\tau \; + s(\eta) \quad (11)$$

Subtracting Eq. (11) from Eq. (10) gives

$$(T\upsilon_R)(\eta) - (T\upsilon_r)(\eta) = \int_0^\eta |k(\eta,\tau)| \left|[F(\upsilon_R(\tau)) - F(\upsilon(\tau))]\right| d\tau$$

Taking the absolute value gives

$$\left|[F(\upsilon_R(\eta)) - F(\upsilon(\eta))]\right| \le \int_0^\eta |k(\eta,t)| \left|[F(\upsilon_R(\tau)) - F(\upsilon(\tau))]\right| d\tau$$

Taking the minimum of both sides and using H_1 and H_2

$q(\upsilon_R(\eta), \upsilon_r(\eta)) \le [\frac{Lk^*}{\Gamma(\alpha+1)}] q(\upsilon_R, \upsilon_r)$

Based on the inequality (9) we have

$q(\upsilon_R(\eta), \upsilon_r(\eta)) \le q(\upsilon_R, \upsilon_r)$

Bases on Banach contraction principle, we can conclude that T has a unique fixed point.

Numerical Examples and Results

Example 1 **[**6**]:** Consider the following Volterra IDE:

$$\upsilon(\eta) - \int_0^{\eta}(\tau - \eta)\upsilon(\tau)d\tau = 1 - \eta - \frac{\eta^2}{2}, \qquad 0 \le \eta, \tau \le 1,$$

The exact solution is $\upsilon(\eta) = 1 - \sinh(\eta)$.

Solving example 1 using the above method at r= 6, 9 and 11. We have the following approximate solutions and the table of results.

$\upsilon_6(\eta) = 1 - 0.9999966196\eta - 0.0000494937\eta^2 - 0.1663944584\eta^3 - 0.00073371301\eta^4 - 0.007298202142\eta^5 - 0.0007286992020\eta^6$

$\upsilon_9(\eta) = 1.000000001 - 1.000000020\eta + 3.961984813 \times 10^{-7}\eta^2 - 0.1666699114\eta^3 + 0.00001349241166\eta^4 - 0.008364106155\eta^5 + 0.00003846318316\eta^6 - 0.0002225010677\eta^7 + 0.000005239899755\eta^8 - 0.000002247601003\eta^9$

$\upsilon_{11}(\eta) = 1 - 1.000000135\eta + 0.000004122675466\eta^2 - 0.1667176617\eta^3 + 0.0003433670968\eta^4 - 0.009747532548\eta^5 + 0.003763918732\eta^6 - 0.006810254907\eta^7 + 0.007619617026\eta^8 - 0.005544705747\eta^9 + 0.002306776509\eta^{10} - 0.0004187054734\eta^{11}$

Example 2 **[**6**]:** Consider the following Volterra IDE:

$$\upsilon(\eta) - \int_0^{\eta}(\tau - \eta)\upsilon(\tau)d\tau = 1, \qquad 0 \le \eta, \tau \le 1,$$

The exact solution is $\upsilon(\eta) = \cos(\eta)$.

Solving example 1 using the above method at r= 6, 9 and 11. We have the following approximate solutions and the table of results.

$\upsilon_6(\eta) = 0.9999999997 - 0.00000127546\eta - 0.4999815547\eta^2 - 0.00009884134\eta^3 + 0.04192088068\eta^4 - 0.000326172424\eta^5 - 0.001210729218\eta^6$

$\upsilon_9(\eta) = 1.000000001 - 2.457952232 \times 10^{-7}\eta - 0.4999945405\eta^2 - 0.00004794297716\eta^3 + 0.04188808319\eta^4 - 0.0005966606799\eta^5 - 0.0004178632255\eta^6 - 0.0009397326475\eta^7 + 0.0005226835316\eta^8 - 0.0001114750437\eta^9$

$\upsilon_{11}(\eta) = 0.9999999994 - 3.213672451 \times 10^{-8}\eta - 0.5000008215\eta^2 + 0.00001133864196\eta^3 + 0.04156623593\eta^4 + 0.0005507466410\eta^5 - 0.003259618314\eta^6 + 0.003994747793\eta^7 - 0.005340866793\eta^8 + 0.004397238022\eta^9 - 0.002008302056\eta^{10} + 0.0003915752299\eta^{11}$

Example 3 **[**6**]:** Let's consider the Fredholm integral equation.

$$\upsilon(\eta) - \int_0^{1}\left(\sqrt{\eta} + \sqrt{\tau}\right)\upsilon(\tau)d\tau = 1 + \eta, \qquad 0 \le \eta, \tau \le 1,$$

The exact solution is $\upsilon(\eta) = -\frac{129}{70} - \frac{141}{35}\sqrt{\eta} + \eta$.

Solving example using the above method at r= 11 and 14. We have the following approximate solutions and the table of results.

$\upsilon_9(\eta) = 1.893166091 - 21.02394029\eta + 156.2850499\eta^2 - 889.5904668\eta^3 + 3877.764724\eta^4 - 13227.01075\eta^5 + 33261.11247\eta^6 - 57645.11698\eta^7 + 65442.76599\eta^8 - 45984.63645\eta^9 + 18019.20874\eta^{10} - 2992.712684\eta^{11}$

$\upsilon_{11}(\eta) = 1.847945499 - 29.37206075\eta + 400.7400768\eta^2 - 4097.141284\eta^3 + 28744.77874\eta^4 - 1.4236230 \times 10^{-5}75 \times 10^{-5}\eta^5 + 5.108223289\eta^6 + 1.349386052 \times 10^{-6}\eta^7 + 2.641142368 \times$

$10^{-6}\eta^8 - 3.818789194 \times 10^{-6}\eta^9 + 4.022427813 \times 10^{-6}\eta^{10} - 2.998134059 \times 10^{-6}\eta^{11} + 1.497143917 \times 10^{-6}\eta^{12} - 4.489689657 \times 10^{-5}\eta^{13} + 61082.11288\eta^{14}$

Results and Discussions

Table 1. Shows a comparison of the Approximate Solution (AS) for example 1

η_i	Exact Solution (ES)	AS $r = 6$	AS $r = 9$	AS $r = 11$
0.0	1.00000000000000	1.00000000000000	1.00000000100000	1.00000000000000
0.2	0.79866399750000	0.79866398480000	0.79866399830000	0.79866399740000
0.4	0.58924767420000	0.58924768640000	0.58924767490000	0.58924767420000
0.6	0.36334641790000	0.36334641190000	0.36334641760000	0.36334641790000
0.8	0.11189401780000	0.11189403780000	0.11189401860000	0.11189401760000
1.0	-0.17520119400000	-0.17520118600000	-0.17520119350000	-0.17520119330000

Table 2. Presents the Absolute Error (AE) comparison for Example 1.

τ_i	AE $r = 6$	AE $r = 9$	AE $r = 11$	[6] AE $r = 10$
0.0	0.000E+00	1.000E-09	0.000E+00	0.000E+00
0.2	1.282E-08	7.569E-10	9.598E-11	2.070E-09
0.4	1.228E-08	6.618E-10	3.807E-11	8.116E-09
0.6	6.035E-09	1.085E-10	2.317E-11	1.765E-08
0.8	1.998E-08	7.770E-10	1.377E-10	2.990E-08
1.0	7.946E-09	3.985E-10	6.755E-10	4.384E-08

Table 3. Shows a comparison of the AS for example 2

τ_i	ES	AS $r = 6$	AS $r = 9$	AS $r = 11$
0.0	1.00000000000000	0.99999999970000	1.00000000100000	0.99999999940000
0.2	0.98006657780000	0.98006658320000	0.98006657840000	0.98006657710000
0.4	0.92106099400000	0.92106099040000	0.92106099550000	0.92106099360000
0.6	0.82533561490000	0.82533562020000	0.82533561540000	0.82533561430000
0.8	0.69670670930000	0.69670670470000	0.69670671010000	0.69670670820000
1.0	0.54030230590000	0.54030230740000	0.69670671010000	0.54030230550000

Table 4. Shows a comparison of the AE for example 2

τ_i	AE $r = 6$	AE $r = 9$	AE $r = 11$	[6] AS $r = 10$
0.0	3.000E-10	1.000E-09	6.000E-10	0.000E+000
0.2	5.447E-09	1.350E-09	1.040E-09	2.809E-008
0.4	3.663E-09	1.090E-09	2.400E-10	5.815E-008
0.6	5.357E-09	1.000E-09	8.000E-10	9.226E-008
0.8	4.520E-09	8.000E-10	1.000E-09	1.327E-007
1.0	1.398E-09	1.000E-09	1.000E-09	1.823E-007

Table 3. Shows a comparison of the AS and AE for example 3

τ_i	ES	AS $r = 11$	AS t $r = 14$	AE $r = 11$	AE $r = 14$	[6] $r = 10$
0.0	-1.842857143	-1.8931660910	-1.847945499	5.031E-02	5.088E-03	8.932E-03
0.2	-3.444489056	-3.4548688450	-3.451315959	1.038E-02	6.827E-03	1.341E-02
0.4	-3.990749429	-4.0150781410	-3.998390709	2.433E-02	7.641E-03	1.597E-02
0.6	-4.363375153	-4.3686125310	-4.371580199	5.237E-03	8.195E-03	1.754E-02
0.8	-4.646120970	-4.6643031910	-4.654883299	1.818E-02	8.838E-03	1.883E-02
1.0	-4.871428572	-4.8474589910	-4.881012699	2.397E-02	8.838E-03	-

AE = Absolute Error, AS = Approximate Solution.

Discussion

The results of applying the Legendre Collocation Method (LCM) to solve IDEs have demonstrated both the accuracy and efficiency of the proposed numerical approach. In the provided examples, we have demonstrated that the method yields approximate solutions that closely match the exact solutions, thereby validating the robustness of the LCM in handling various types of IDEs. In example 1, the approximate solutions obtained using different orders of shifted Legendre polynomials (r = 6, 9, 11) were compared to the exact solution. The results show that as the order of the polynomial increases, the approximation becomes more accurate. This is evident from the comparison of the approximate solutions with the exact solution in Table 1, where the solutions for r=9 and r=11 closely match the exact solution across all evaluated points.

The absolute errors presented in Table 2 further support the effectiveness of the method. The errors decrease as the polynomial order increases, demonstrating the convergence of the LCM to the exact solution. The maximum absolute error for r=11 was found to be in the order of 10^{-9}, indicating a highly precise approximation.

In the second example, the approximate solutions for r=6, 9, and 11 were again evaluated. The results, shown in Table 3, indicate that the approximations are highly accurate, with the polynomial order r=11 providing the closest match to the exact solution.

The absolute errors in Table 4 demonstrate that the LCM effectively reduces errors as the order of the polynomial increases, with the maximum error for r=11 being as low as 10^{-9}. This reaffirms the high accuracy and convergence properties of the LCM.

For the third example, with the exact solution, the approximate solutions for r=9 and r=11 were computed. The approximate solutions presented in the example show that the LCM can handle both Volterra and Fredholm types of IDEs effectively, providing accurate results with increasing polynomial order.

Overall, the numerical examples validate that the Legendre Collocation Method is a robust and accurate tool for solving various types of integro-differential equations. The method's ability to provide highly accurate approximations, as demonstrated by the low absolute errors and close match with exact solutions, makes it a valuable technique for researchers and practitioners dealing with complex IDEs.

Conclusion

The study has presented the Legendre Collocation Method (LCM) as an effective numerical technique for solving integro-differential equations (IDEs). By leveraging shifted Legendre polynomials to construct approximate solutions, the method transforms IDEs into systems of linear algebraic equations, which can be solved to obtain highly accurate solutions. The numerical examples provided in this study showcase the method's effectiveness in handling both Volterra and Fredholm types of IDEs.

The results indicate that the LCM offers several advantages:

1. **High Accuracy:** The method provides solutions that closely match exact solutions, with low absolute errors even for higher-order polynomials.
2. **Convergence:** The LCM demonstrates strong convergence properties, with errors decreasing as the polynomial order increases.
3. **Robustness:** The method effectively handles different types of IDEs, including those with complex integral and differential terms.

In conclusion, the Legendre Collocation Method is a powerful and reliable tool for solving integro-differential equations. Its ability to provide accurate approximations makes it a valuable addition to the numerical techniques available for tackling complex mathematical models in various scientific and engineering fields. Future work may focus on extending the method to more generalized forms of IDEs and exploring its application in real-world problems where IDEs play a crucial role.

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Conflicts of interest

This statement serves to declare that the authors have no conflicts of interest regarding the work submitted for publication consideration in your journal, *Journal of Natural Sciences and Mathematics Research.*

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