Journal of Natural Sciences and Mathematics Research
J. Nat. Scien. & Math. Res. Vol. 11 No. 2 (2025) 113-121
ISSN: 2460-4453 (Online); 2614-6487 (Printed)
Available online at http://journal.walisongo.ac.id/index.php/jnsmr

Null Bézier Curves in Minkowski 3-Space

Arfah Arfah[1,2*]
[1] Department of Mathematics, State University of Makassar, Indonesia.
[2] Department of Mathematics, karedeniz Technical University, Turkey.

* Corresponding author's e-mail: arfah@unm.ac.id

ABSTRACT
In this paper, we define and investigate the properties of null Bézier curves in Minkowski 3-space. The method applied is a theoretical literature study, applying the definitions of Bézier curves and the geometric framework of null curves in semi-Riemannian geometry. We establish several fundamental characteristics of these curves, including the causal nature of their tangent vectors at endpoints and their Frenet frame apparatus when parametrized by pseudo-arc length. Furthermore, we define the concept of a null Bertrand pair for such curves and prove that if a null Bézier curve of degree n≥3 admits a Bertrand mate, then both curves are necessarily helices. Finally, we provide a conclusive parametric representation of any null Bézier curve in terms of a single non-constant function. This representation offers a powerful tool for explicitly constructing null Bézier curves within this geometric setting.
Keywords:
Null Bézier curves; Bertrand pair curves; representation function; Minkowski space.
Subject Classification Code: 53C50, 14H05, 14H50, 53A35.

Introduction

Since the second half of the twentieth century, the theory of differential geometry in semi-Riemannian space has been active area of research in differential geometry due to its applications in various fields especially in physics. For instance, physicists study gravity and general relativity by applying the theory of differential geometry in Lorentz-Minkowski space. Comprehensive study about differential of curves and surfaces in Minkowski space can be seen in (López, 2014). Based on their tangent vectors, curves in Minkowski space are categorized into three types, i.e., spacelike, timelike and null curves. Properties of non-null curves can generally can be derived from those of curves in a Riemannian manifold. However, the geometry of null curves is different since the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. As a consequence, a different approach is needed to study null curves. Some studies about null curves can be seen in some references (Duggal & Jin, 2007; Ferrández et al., 2001; Inoguchi & Lee, 2008; Liu & Jung, 2017; Sakaki, 2010).

Bézier curves were developed by Pierre Bézier in the 1960 at Renault Automobile, who devised a mathematical method of curves for the drawing of technical drawing in aeronautical and mechanical design (Rogers, 2001). Bézier curves and surfaces are widely used in Computer Aided Geometric Design (CAGD) and Computer Graphics (CG) with given control points and are formed uniquely by the Bernstein basis functions. As a consequence, the concepts of Bézier curves have been applied in many areas of science, technology and medicine. Fierz in (Fierz, 2018) developed a method to calculate likelihood ratios for plasma amyloid-beta biomarkers for Alzheimer's disease by using Bézier curves. Other applications of Bézier curves can be seen in some references (Deslierres, 1998; Harada & Nakamae, 1982). There have been many books and research articles which can be used to study about Bézier curves (Buss, 2003; Goldmand, 2019; Marsh, 2005; Yang & Zeng, 2009). In Minkowskian space, (Samanci et al., 2019; Samancı, 2018) have studied the timelike and spacelike Bézier curves. We give some basic information about the Bézier curves. Then we define the null Bézier curves and some corollaries. We also define the null Bertrand pair of null Bézier curves and show that both curves must be helices. Next, we give the parametric representation of a null Bézier curve in terms of a non-constant function.

Methods

The method of this research is literature study. Research begins with study about the theory of the Bézier curve and the geometric properties of the semi-Riemann manifold, especially about the Null curve in Minkowski space. Next, applying the basic definitions, and the existing results from the related work, we build the definition of the null Bézier curves in Minkowski $\mathbb{R}_1^3$ and investigate this new type of curve.

In this section, we provide some definitions, theorems and notations that underlie the main results.

Let (R_1^3, g) be a Minkowski 3-space defined by the Lorentzian metric

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are vectors in $(\mathbb{R}_1^3, g)$. Based on their causal characteristics, a vector $v \in (\mathbb{R}_1^3, g)$ is said to be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$, or null if $g(v, v) = 0$ and $v \neq 0$. In Minkowski 3-space, a curve is a spacelike (resp. timelike or null) if the tangent vector along the curve is spacelike (resp. timelike or null).

Let $r: I \to \mathbb{R}_1^3$ be a null curve parametrized by pseudo-arc length s. Suppose T, N, and B denoted the tangent, principal normal, and binormal vectors of the curve r. The Frenet frame of r can be expressed by

$$T(s) = r'(s),\ N(s) = r''(s),\ B(s) = -r'''(s) - \tfrac{1}{2}g\big(r'''(s), r'''(s)\big)r'(s), \tag{1}$$

where

$$\langle T(s), T(s)\rangle = \langle B(s), B(s)\rangle = \langle T(s), N(s)\rangle = \langle B(s), N(s)\rangle = 0$$
$$\langle T(s), B(s)\rangle = \langle N(s), N(s)\rangle = 1.$$

Therefore, the Frenet formula of curve r is given by

$$T'(s) = r'(s),\ N'(s) = \kappa(s)T(s) - B(s),\ B'(s) = -\kappa(s)N(s), \tag{2}$$

where κ denotes the null curvature of r and is given by

$$\kappa(s) = -\tfrac{1}{2}g\big(r'''(s), r'''(s)\big). \tag{3}$$

(Liu & Jung, 2017)

Bézier curves are polynomial curves which have a particular mathematical representation. A Bézier curve of degree n specified by a sequence of $n + 1$ points which are called control points. The polygon obtained by joining the control points with line segments in the prescribed order is called the control polygon.

Definition 1. Let $n + 1$ control points $b_0, b_1, \dots, b_n$ the Bézier curve of degree n is defined to be

$$B(t) = \textstyle\sum_{i=0}^{n} b_i\, B_{i,n}(t),\ t \in [0,1] \tag{4}$$

where

$$B_{i,n}(t)=\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i,\ \text{ if } 0\le i\le n \tag{5}$$

are called the Berstein polynomials or Bernstein basis functions of degree n.

Bézier curves of dimension 1, 2, and 3 are called linear, quadratic and cubic Bézier curves, respectively. The first derivative of n degree Bézier curve $B(t)$ is defined by

$$B'(t)=\sum_{i=0}^{n} b_i\, B'_{i,n}(t),\ \ t\in[0,1]. \tag{5}$$

For more general, the rth derivative of the Bézier curve denoted by $B^{(r)}$ is given by

$$\frac{d^rB(t)}{dt^r}=\frac{n!}{(n-r)!}\sum_{i=0}^{n} B_{i,n}\Delta^r b_i, \tag{7}$$

where $\Delta^r b_i=\sum_{i=0}^{r}\frac{r!}{(r-j)!}(-1)^{r-j}b_{i+j}$ and $\Delta^r b_i=\Delta^{r-1}b_{j+1}-\Delta^{r-1}b_j$ for $b_i\in\mathbb{R}_1^3$ (Samancı, 2018).

Result and Discussion

Definition 2. The Bézier curve of degree n $B(t)=\sum_{i=0}^{n} b_i\, B_{i,n}(t)$ for $b_i\in\mathbb{R}_1^3, i=0,\dots,n$ and $t\in[0,1]$ is called a null Bézier curve if the tangent vector $\boldsymbol{B}'(t)$ is a null vector for all $t\in[0,1]$ in minkowski space $\mathbb{R}_1^3$.

It follows that in the cases $n=1, n=2$, and $n=3$ correspond to linear, quadratic, and cubic null Bézier curves, respectively.

Definition 3. Let $b_0,b_1,\dots,b_n\in\mathbb{R}^3$ and $B(t)=\sum_{i=0}^{n} b_i\, B_{i,n}(t)$ is null Bézier curve. The first order derivative of $B(t)$ is called hodograph. If $\Delta b_i=b_{i+1}-b_i$ for $i=0,1,\dots,n-1$ are null vectors in the same cone then the hodograph curve of $B(t)$ is a null curve in the same cone. We call this curve null hodograph curve.

Theorem 1. *Let $B(t)$ is a null Bézier curve of degree n. The the first derivative of $B(t)$ can be expressed as*

$$B'(t)=n\sum_{i=0}^{n}\left(B_{i-1,n-1}(t)-B_{i,n-1}(t)\right)b_i,\ \ t\in[0,1] \tag{8}$$

Proof. Consider the null Bézier curve

$$B(t)=\sum_{i=0}^{n} b_i\, B_{i,n}(t),\qquad t\in[0,1],$$

where

$$B_{i,n}(t)=\frac{n!}{(n-i)!\,i!}(1-t)^{n-i}t^i,\ \ if\ 0\le i\le n.$$

Then,

$B'_{i,n}(t) = \frac{n!}{(n-i)!i!}\left(-(n-i)(1-t)^{n-i-1}t^i + i(1-t)^{n-i}t^{i-1}\right)$

$= n\left(B_{i-1,n-1}(t) - B_{i,n}(t)\right).$

Therefore,

$$B'(t) = \frac{bB(t)}{dt} = \sum_{i=0}^{n} b_i\, B'_{i,n}(t) = \sum_{i=0}^{n} \left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right) b_i$$

for $t \in [0,1]$.

Theorem 2. *Let $B(t)$ be a Bézier curve of degree n. The first derivative of $B(t)$ can also* be expressed as

$$B'(t) = \sum_{i=0}^{n} n\, B_{i,n}(t)\Delta b_i,$$

where $\Delta b_i = b_{i+1} - b_i$.

Proof. From equation (8) and using the fact that $B_{-1,n-1}(t) = B_{n,n-1}(t) = 0$, we obtain

$$B'(t) = \sum_{i=1}^{n} n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right) b_i$$

$$= \sum_{i=1}^{n} nB_{i-1,n-1}(t)\, b_i - \sum_{i=0}^{n} nB_{i,n-1}(t)\, b_i.$$

Renumbering the first summation gives

$$B'(t) = \sum_{i=0}^{n-1} nB_{i,n-1}(t)\, b_i - \sum_{i=0}^{n-1} nB_{i,,n-1}(t)\, b_i$$

$$= \sum_{i=0}^{n-1} B_{i,n-1}(t)\, \Delta b_i.$$

Theorem 3. *Let $B(t)$ be a null Bézier curve of degree n. The tangent vectors at the initial and end points are null vectors.*

Proof. Let $B(t)$ be a null Bézier curve, then

$$\frac{dB(t)}{dt} = n\ \cdot \Delta b_0$$

for $t = 0$. It implies that $g(\Delta b_0, \Delta b_0) = 0$ and hence, Δb_0 is a null vector. Similarly, if we set $t = 1$, we have

$$\frac{dB(t)}{dt} = n\ \cdot \Delta b_{n-1}.$$

Since $g(\Delta b_{n-1}, \Delta b_{n-1}) = 0$ then Δb_{n-1} is also a null vector, and this completes the proof.

Next, we show some causal characteristics of null Bézier curves parametrized by pseudo-arc length in Minkowski 3-space. Therefore, we first introduce the pseudo-arc length as follows.

Definition 4. *Let $B(t)$ be a null Bézier curve of degree n. We say $B(t)$ is parametrized by the pseido-arc length iff $g\big(B''(t), B''(t)\big) = 1$ and*

$$u = u(t) = \int_{t_0}^{t} \Big(\big(B''(t), B''(t)\big)\Big)^{1/4} dt$$

is called a pseudo-arc length parameter.

Let $\alpha(u), \beta(u)$, and $\gamma(u)$ be respectively the tangent, principal normal, and binormal vector field of null Bézier curve $B(u)$ in Minkowski 3-space. Then, we have the following theorems.

Theorem 4. *Let $B(u)$ be Bézier curves of degree n in Minkowski 3-spaces parametrized by the pseudo arc-length u. Then there exists a Frenet frame $\{\alpha, \beta, \gamma\}$ satisfying the following equations*

$$\alpha = n \sum_{i=0}^{n-1} B_i^{n-1}(u)(b_{i+1} - b_i)$$

$$\beta = (n^2 - n) \sum_{i=0}^{n-2} B_i^{n-2}(u)(b_i - b_{i+1} + b_{i+2})$$

$$\gamma = (n^3 - 3n^2 + (2 + \kappa)n) \sum_{i=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})$$
$$+\kappa n(n-1)(1-u)^{n-2} b_{n-2} + \kappa\big((n-1)u^{n-2} - nu^{n-1}\big)b_{n-1} + \kappa u^{n-1} b_n,$$

where the null curvature κ of $B(u)$ is given by

$$\kappa = -\frac{1}{2}(n^3 - 3n^2 + 2n)^2 \left\|\sum_{I=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})\right\|. \tag{9}$$

Proof. Using equations (1) and (3), we get

$$\gamma = (n^3 - 3n^2 + 2n) \sum_{I=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})$$
$$+\kappa n \sum_{I=0}^{n-1} B_i^{n-i}(u)(b_{i+1} - b_i)$$
$$= (n^3 - 3n^2 + (2 + \kappa)n) \sum_{I=0}^{n-3} B_i^{n-3}(u)(-2b_i + 4b_{i+1} - 3b_{i+2} + b_{i+3})$$
$$+\kappa n(B_{n-2}^{n-1}(u)(b_{n-1} - b_{n-2}) + \kappa n \left(B_{n-2}^{n-1}(u)(b_n - b_{n-1})\right)$$
$$= (n^3 - 3n^2 + (2 + \kappa)n) \sum_{I=0}^{n-3} B_i^{n-3}(u)(-2b_i + 4b_{i+1} - 3b_{i+2} + b_{i+3})$$
$$-\kappa n(n-1)(1-u)u^{n-2} b_{n-2} b_{n-2} + \big((n-1)u^{n-2} - nu^{n-1}\big)b_{n-1} + u^{n-1} b.$$

This completes the proof.

Theorem 4 implies the following corollaries as follow.

Corollary 1. *The linear and quadratic null Bézier curves in Minkowski 3-space are plane curves.*

Proof. Let $B(u)$ be a null Bézier curve of in Minkowski 3-space. If $n = 1$ or $n = 2$, by the curvature κ given in Theorem 4, we have $\kappa = 0$, and hence $B(u)$ is a plane curve.

Corollary 2. *Let $B(u)$ be a null Bézier curve of degree $n \geq 3$. Then $B(u)$ is generalized helix iff* $\left\|\sum_{I=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})\right\| = 0$.

Definition 5. *Let $B(u)$ and $\bar{B}(\bar{u})$ are null Bézier curves parametrized by u and $\bar{u}$ in Minkowski 3-space. The pair $\left(B(u), \bar{B}(\bar{u})\right)$ is called a null Bertrand pair iff the principal normal vector fields are linearly dependent.*

Theorem 5. *Let $B(u)$ and $\bar{B}(u)$ are null Bézier curves of degree $n \geq 3$ in parametrized by u and $\bar{u}$ in Minkowski 3-space. If $B(u)$ admits a Bertrand mate $\bar{B}(\bar{u})$ then*

$$\left\|\sum_{I=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})\right\| = constant.$$

Proof. Suppose $B(u)$ and $\bar{B}(\bar{u})$ are null Bézier curves parametrized by pseudo arc-length u and $\bar{u}$. Since $B(u)$ admits a Bertrand mate $\bar{B}(\bar{u})$ then their normal lines coincide at corresponding points. As a consequence, we may write

$$\bar{B} = B + \eta\beta, \tag{10}$$

for some function $\eta(p) \neq 0$. Differentiating the equation (10) with respect to u and using equation (2), we get

$$\frac{d\bar{u}}{du}\bar{\alpha} = (1 + \eta\kappa)\alpha - \eta\gamma + \eta'\beta \tag{11}$$

where $\bar{\alpha}$ is the tangent vector of $\bar{B}$ and α, β, γ are the tangent, principal normal, and binormal vector fields of B. For null Bertrand curves, the condition $g(\bar{\alpha}, \gamma) = 0$ holds. Therefore, from equation (11), we have $\eta' = 0$ and so η is a constant. Since $\bar{\alpha}$ is a null vector, taking the norm of the equation (11) yields $\eta(1 + \eta\kappa) = 0$ and so $\kappa = -\frac{1}{\eta} = constant$. Hence, by equation (9), we get $\left\|\sum_{I=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})\right\| = constant$ for $n \geq 3$.

Theorem 6. *Let $B(u)$ and $\bar{B}(u)$ are null Bézier curves of degree $n \geq 3$ in parametrized by u and $\bar{u}$ in Minkowski 3-space. If $B(u)$ admits a Bertrand mate $\bar{B}(\bar{u})$ then*

$$\bar{\alpha} = \mp\frac{(n^3 - 3n^2 + (2 + \kappa)n)}{\kappa}\sum_{I=0}^{n-3} B_i^{n-3}(u)(-b_i + 3b_{i+1} - 3b_{i+2} + b_{i+3})$$

$$\mp n(n-1)(1-u)u^{n-2}b_{n-2} \pm \left((n-1)u^{n-2} - nu^{n-1}\right)b_{n-1} \mp u^{n-1}b_n,$$

$$\bar{\beta} = -(n^2 - n)\sum_{I=0}^{n-2} B_i^{n-2}(b_i - b_{i+1} + b_{i+2}),$$

$$\bar{\gamma} = \mp\kappa n\sum_{I=0}^{n-3} B_i^{n-1}(u)(b_{i+1} - b_i),$$

where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are the tangent, principal normal, dan binormal vector fields of $\bar{B}$ and κ is the null curvature of B.

Proof. From Theorem 5, we find $\eta = -\frac{1}{\kappa} = constant$. As a result, if we substitute this into equation (11), we find

$$\bar{\alpha} = \mu\gamma, \; \mu = -\eta\frac{du}{d\bar{u}}. \tag{12}$$

Differentiating equation (12) with respect to u and using equation (2) yield

$$\frac{d\bar{u}}{du}\bar{\beta} = \mu'\gamma - \mu\kappa\beta. \tag{13}$$

Next, taking the norm of equation (13) gives

$$\left(\frac{d\bar{u}}{du}\right)^2 = \mu^2\kappa^2.$$

Using equations (12) and (13), we get $\frac{d\bar{u}}{du} = \pm 1$ and $\mu = \pm\eta = constant$. As a consequence, we find $\bar{\alpha} = \mp\frac{1}{\kappa}$, $\bar{\beta} = -\beta$ and $\bar{\gamma} = \mp\kappa\alpha$. Substituting α, β, and γ from equation Theorem 4 completes the proof.

Corollary 3. *Let $B(u)$ and $\bar{B}(u)$ be null Bézier curves of degree $n \geq 3$ in parametrized by u and $\bar{u}$ in Minkowski 3-space. If $B(u)$ admits a Bertrand mate $\bar{B}(\bar{u})$ then their null curvatures are equal.*

From Theorem 6. we have $\bar{\gamma} = \mp\kappa\alpha$. Differentiating this gives $\bar{\gamma}' = -\kappa\beta$. It follows that $\bar{\kappa} = g(\bar{\gamma}', \bar{\gamma}) = g(-\kappa N, -N) = \kappa$.

Corollary 4. *Let $B(u)$ and $\bar{B}(u)$ be null Bézier curves of degree $n \geq 3$ in parametrized by u and $\bar{u}$ in Minkowski 3-space. If $B(u)$ admits a Bertrand mate $\bar{B}(\bar{u})$ then both are helices.*

Proof. It follows from the fact that κ and $\bar{\kappa}$ are constants.

Next, we define the null Bézier curve using a representation function. Let $B(u)$ be a null Bézier curve in Minkowski 3-space with the pseudo arc-length parameter u. By Theorem 2, we can write

$$B'(u) = \left(\sum_{i=0}^{n} nB_i'^{n}(u)\Delta x_i, \sum_{i=0}^{n} nB_i'^{n}(u)\Delta y_i, \sum_{i=0}^{n} nB_i'^{n}(u)\Delta z_i\right),$$

where $b_i = (x_i, y_i, z_i), i = 0,1,2,\dots,n$.

Now let $\sum_{i=0}^{n} nB_i'^{n}(u)\Delta x_i = \lambda_1(u)$, $\sum_{i=0}^{n} nB_i'^{n}(u)\Delta y_i = \lambda_2(u)$, $\sum_{i=0}^{n} nB_i'^{n}(u)\Delta z_i = \lambda_3(u)$, so that $B'(u) = \big(\lambda_1(u), \lambda_2(u), \lambda_3(u)\big)$.

Consequently, we have

$$-\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0.$$

It follows that

$$\frac{\lambda_3 + \lambda_1}{\lambda_2} = \frac{\lambda_2}{\lambda_1 - \lambda_3}.$$

Without loss of generality, we may assume that

$$\frac{\lambda_3+\lambda_1}{\lambda_2} = \frac{\lambda_2}{\lambda_1-\lambda_3} = f(u) \tag{14}$$

and

$$\lambda_2 = 2\phi(u). \tag{15}$$

From equations (14) and (15), we have

$$\lambda_1 + \lambda_3 = 2\phi f, \qquad \lambda_2 = 2\phi, \qquad \lambda_1 - \lambda_3 = \frac{2\phi}{f}.$$

Hence, solving this system of equations gives us

$$\lambda_1 = \phi\left(f + \frac{1}{f}\right), \qquad \lambda_2 = 2\phi, \qquad \lambda_3 = \phi\left(f - \frac{1}{f}\right).$$

It follows that

$$B'(u) = \phi(u)\left(f(u) + \frac{1}{f(u)}, 2, f(u) - \frac{1}{f(u)}\right).$$

On the other hand, null Bézier curves can also be written as

$$B(u) = \int B'(u)du = \int \phi(u)\left(f(u) + \frac{1}{f(u)}, 2, f(u) - \frac{1}{f(u)}\right). \tag{16}$$

Differentiating equation (4.3) with respect to u twice gives

$$g\big(B''(u), B''(u)\big) = 4\phi^2(u)\left(\frac{f'(u)}{f(u)}\right)^2.$$

Since u is the pseudo arc-length parameter of the null Bézier curve $B(t)$, then we have

$$\phi(u) = \pm\frac{f(u)}{2f'(u)}.$$

This result can be summarized as the following theorem.

Theorem 7. *Let B(u) be a null Bézier curve in Minkowski 3-space with the pseudo arc-length parameter u. Then $B(u)$ can be written as*

$$B(u) = \int \pm\frac{f(u)}{2f'(u)}\left(f(u) + \frac{1}{f(u)}, 2, f(u) - \frac{1}{f(u)}\right)du$$

for some nonconstant function $f(s)$.

Conclusion

In this paper, we have undertaken a comprehensive study of Bézier curves within the specific context of null curves in Minkowski 3-space. By integrating the theory of Bézier curves with the unique geometric structure of semi-Riemannian manifolds, we have successfully defined and characterized null Bézier curves. The primary contributions of this work are threefold. First, we established fundamental definitions and theorems for null Bézier curves, demonstrating key properties such as the null nature of their initial and end tangent vectors. Second, we introduced and analyzed the concept of a null Bertrand pair for these curves. Our results show that for a null Bézier curve of degree $n \geq 3$ to admit a Bertrand mate, its pseudo-torsion must be constant, which in turn implies that both curves in the pair are necessarily helices. Finally, we derived a general parametric representation for any null Bézier curve in terms of a single nonconstant function $f(u)$. This representation provides a powerful and versatile method for the explicit construction and analysis of null Bézier curves, encapsulating their essential geometry in a concise formula.

Acknowledgments

The authors would like to thank the referees for their helpful suggestions.

Conflicts of interest

The authors declare that there are no conflicts of interest.

References

Buss, S. R. (2003). *3D Computer Graphics: A Mathematical Introduction with OpenGL*. University of California.

Deslierres, M. (1998). Bezier Curves for Economic Applications. *Computers in Higher Education Economics Review*, *12*(1), 6–12.

Duggal, K., & Jin, D. H. (2007). *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*. World Scientific.

Ferrández, A., Giménez, A., & Lucas, P. (2001). Null helices in Lorentzian space forms. *International Journal of Modern Physics A*, *16*.

Fierz, W. (2018). Application of Bézier Curves for Calculating Likelihood Ratios for Plasma Amyloid-β Biomarkers for Alzheimer's Disease. *Frontiers in Aging Neuroscience*, *10*.

Goldmand, R. (2019). *An Integrated Introduction to Computer Graphics and Geometric Modeling* (1st ed.). CRC Press.

Harada, K., & Nakamae, E. (1982). Application of the Bézier curve to data interpolation. *Computer-Aided Design*, *4*(1), 55–59.

Inoguchi, J. İ., & Lee, S. (2008). Null curves in Minkowski 3-space. *International Electronic Journal of Geometry*, *1*(2), 40–83.

Liu, H., & Jung, S. D. (2017). Null Curves and Representation in Three Dimensional Minkowski Spacetime. *New Horizons in Mathematical Physics*, *1*(1), 1–10.

López, R. (2014). Differential geometry of curves and surfaces in Lorentz-Minkowski space. *International Electronic Journal of Geometry*, *7*(1), 44–107.

Marsh, D. (2005). *Applied geometry for computer graphics and CAD*. Spinger.

Rogers, D. F. (2001). *An introduction to NURBS with historical perspective*. Morgan Kaufmann.

Sakaki, M. (2010). Notes on null curves in Minkowski spaces. *Turkish Journal of Mathematics*, *34*(3), 417–424.

Samanci, H. K., Kalkan, O., & Celik, S. (2019). The timelike Bezier Spline in Minkowski 3 Space. *Journal of Science and Art*, *2*(47), 357–374.

Samancı, H. K. (2018). Some Geometric Properties of The Spacelike Bezier Curve with a Timelike Principal Normal in Minkowski 3-Space. *Cumhuriyet Science Journal*, *39*(1), 71–79.

Yang, L.-Q., & Zeng, X.-M. (2009). Bézier curves and surfaces with shape parameter. *International Journal of Computer Mathematics*, *86*(7), 1253–1263.