Journal of Natural Sciences and Mathematics Research
J. Nat. Scien. & Math. Res. Vol. 11 No. 1 (2025) 74-83
ISSN: 2460-4453 (Online); 2614-6487 (Printed)
Available online at http://journal.walisongo.ac.id/index.php/jnsmr

Morrey spaces in quantum theory: on regularity of the solution of Schrödinger equation via fractional maximal operators

Yusuf Ramadana
Department of Mathematics, State University of Makassar, Indonesia.

Corresponding author's e-mail: yusuf.ramadana@unm.ac.id

ABSTRACT
In this paper, we investigate the regularity conditions of solutions of Schrödinger Equations with forcing function in the framework of generalized weighted Morrey spaces. We utilize the established boundedness of the fractional maximal operator on generalized weighted Morrey spaces. We prove that if the forcing function does not balance local regularity and global decay, then the solution of the equation does not do so either.
Keywords: Maximal fractional operator, Morrey spaces, Quantum theory, Schrödinger equations, Schrödinger operator.

Introduction

Quantum mechanics is one of a brilliant outcomes of thinking in recent decades. The concept extends the physics subject. Quantum theory provides a foundational framework for understanding physical phenomena at microscopic scales, including the behavior of subatomic particles and the structure of matter. Within this framework, analyzing partial differential equations, such as the Schrödinger and Klein–Gordon equations, is crucial for revealing the fundamental dynamics of quantum systems. An important part of this analysis is investigating the regularity of solutions, which offers valuable insights into the stability and overall behavior of quantum states. Quantum theory began with the work of Max Planck in the early 20th century when he studied the blackbody radiation (Longair, 2020). Max Planck won the Nobel Prize for the work.

Consider the Schrödinger Equation

$$[V^{\gamma}(-\Delta+V)^{-\beta}](u)=f \tag{1}$$

where $0\le\gamma\le\beta\le 1$, and the equation

$$[V^{\gamma}\nabla(-\Delta+V)^{-\beta}](u)=f \tag{2}$$

where $0\le\gamma\le\frac{1}{2}\le 1, \beta-\gamma\ge\frac{1}{2}$, and V is a nonnegative potential belonging to the reverse Hölder class which is denoted by B_∞.

The function f in equation (1) and (2) is called data, dumping function, source term, or forcing function according to the context. f represents the external forces influencing the system. In this work, we choose to refer to it as the forcing function.

The equations describe the behaviour of particle in a quantum space. One may see the recent works by Geng et al (2023), Hossein et al (2023), Ibrahim et al (2023), Ibrahim & Baleanu (2023), Litu et al (2023), and Rafiq et al (2023) for more about the Schrödinger equation. The equation is difficult to solve in explicit form for u. However, we can still learn the properties of the solution using some tools from mathematical analysis. In this paper, we are interested in answering the question of *"what is happen to the solution u if the data f in equation (1) and (2)* *does not balance local regularity and global decay?".* In the other words, we aim to examine the characteristics of the solution u when neither the local nor global behavior of f can not be controlled.

The notions of local and global behavior can be characterized in terms of Morrey spaces. The concept of Morrey spaces originated from Charles Bradfield Morrey's work in 1938, which focused on the local properties of solutions to specific elliptic partial differential equations (Morrey, 1938). Morrey spaces can be regarded as one of several generalizations of Lebesgue spaces. Morrey space $\mathcal{L}_q^p$ is a set that collects any measurable function $f \in L_{\mathrm{loc}}^p$ in which the norm $\|f\|_{\mathcal{L}_q^p}$ which is defined by

$$\|f\|_{\mathcal{L}_q^p} = \sup_{z\in\mathbb{R}^n, l>0} \frac{1}{|B(z,r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(z,l)} |f(y)|^p \, dy\right)^{\frac{1}{p}}$$

is finite. There are some extensions of Morrey spaces, namely Orlicz-Morrey spaces (Kawasumi, 2023; Kawasumi et al, 2023; Shi et al, 2021), generalized Morrey spaces (Nakai, 1991; Mizuhara, 1993; Karapetyants & de Cristoforis, 2023; Sawano, 2019), weighted Morrey spaces (Komori & Shirai, 2009; Duoandikoetxea & Rosenthal, 2021), and generalized weighted Morrey spaces (Ramadana, 2022; Ramadana & Gunawan, 2023; Ramadana & Gunawan, 2024). The spaces were widely used in analyzing the properties of the solution of some partial differential equations via some operators studied from the subject of mathematical analysis, particularly harmonic analysis. The operator that were commonly used are the Hardy-Littlewood maximal operator, fractional integral operator, Calderon-Zygmund operator, Fourier transform, etc. In the general space, the reader may see the work of Ramadana & Gunawan (2024) and Samko (2022) for the boundedness of several operators on the spaces.

In this paper, we study the solution of (1) as well as (2) where f is a function in Morrey space, particularly related to regularity property of the solution. Particularly, we investigate it in general setting, namely generalized weighted Morrey spaces over $\mathbb{R}^n$. We examine the smoothness characteristics of the solutions to Schrödinger equations (1) and (2) by employing harmonic-analysis techniques, particularly the fractional maximal operator M_α. Our approach begins by proving that these operators act boundedly on generalized weighted Morrey spaces. We then use these findings to investigate the behavior of the equations' solutions.

Truong et al (2020) have studied equation (1) within the context of generalized Morrey spaces where $\gamma = 0$. On the other hand, in the present work, we focus on the more general case $0 \le \gamma \le \beta \le 1$. In recent developments, several studies have explored the interaction between fractional maximal operators and Schrödinger-type operators within the framework of Morrey-type spaces. For instance, Shu and Wang (2015) investigated the boundedness of the fractional maximal operator and Marcinkiewicz integrals associated with Schrödinger operators on Morrey spaces with variable exponent, establishing important estimates that generalize classical results to the constant exponent case. Subsequently, Kucukaslan (2022) extended these ideas to the Lorentz–Morrey setting, where maximal and fractional maximal operators were applied to analyze Bochner–Riesz and Schrödinger-type operators, yielding refined boundedness criteria. More recently, Guliyev et al (2024) examined fractional integrals related to Schrödinger operators on vanishing generalized mixed Morrey spaces, providing new insights into compactness and decay properties of these operators. Collectively, these studies underscore the growing significance of Morrey-type frameworks in understanding the mapping and regularity properties of fractional and Schrödinger-type operators.

The use of Morrey spaces in the present work is motivated by their crucial role in describing the fine local behavior of functions that arise as solutions to partial differential equations, particularly the Schrödinger equation. Unlike the classical Lebesgue spaces, Morrey spaces simultaneously measure the integrability and the concentration of a function, providing an intermediate scale between local and global regularity. This dual nature makes them a natural framework for analyzing the boundedness of fractional maximal operators, which are closely connected to potential-type estimates in quantum theory.

In the context of the Schrödinger equation, Morrey spaces allow one to capture the delicate balance between oscillation and decay of the wave function, particularly when dealing with singular or fractional potentials. The boundedness of M_α on Morrey spaces plays a crucial role in deriving regularity and embedding results, offering a refined understanding beyond what can be obtained in the standard L^p setting. Possible extensions of the present study include the use of generalized Morrey, variable exponent, Herz–Morrey, or Orlicz–Morrey spaces, which may provide further flexibility in handling anisotropic potentials or nonlocal operators that appear in modern quantum models.

Prelimanaries: Some Definitions & Notations

Generalized weighted Morrey spaces were introduced by Guliyev (2014), Guliyev & Ismailova (2014), and Nakamura (Nakamura, 2016) with his another version of the definition. In this paper, we use the definition by Guliyev. Before we define state the definitions, we need some preliminaries on some other spaces and parameters.

The measure and integral used in this paper are Lebesgue measure and integration. $B(a,r)$ is the ball on $\mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ with radius of $r > 0$. We denote E^c by the complement of the subset $E \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$. A weight w is a nonnegative (almost everywhere) which is locally integrable function defined on the Euclidean space $\mathbb{R}^n$. For $E \subseteq \mathbb{R}^n$ we write $w(E) = \int_E w(y)dy$. In addition, $|E|$ denotes the usual Lebesgue measure of the measurable subset E of $\mathbb{R}^n$.

Let $p \in (1,\infty)$. We denote p' as a positive real number which satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. The weighted Lebesgue space $L^{p,w}(E)$ over E is a space that collects any function f on $\mathbb{R}^n$ such $\|f\|_{L^{p,w}(E)} < \infty$ where

$$\|f\|_{L^{p,w}(E)} = \left(\int_E |f|^p w\right)^{\frac{1}{p}}.$$

If $E = \mathbb{R}^n$, we write $L^{p,w} = L^{p,w}(\mathbb{R}^n)$. We now review the definitions of the Muckenhoupt weight classes A_p and $A_{p,q}$. A weight w belongs to A_p if

$$\sup_B \left(\frac{1}{|B|}\int_B w(y)\,dy\right)^{\frac{1}{p}} \left(\frac{1}{|B|}\int_B w(y)^{\frac{p'}{p}}\,dy\right)^{\frac{1}{p'}} < \infty$$

with the supremum evaluated over the entire collection of balls B on $\mathbb{R}^n$. For $1 < p \le q < \infty$, the weight w is in $A_{p,q}$ if

$$\sup_B \left(\frac{1}{|B|}\int_B w^q(y)\,dy\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B w(y)^{p'}\,dy\right)^{\frac{1}{p'}} < \infty$$

with the supremum evaluated over the entire collection of balls B on $\mathbb{R}^n$. If $w \in A_{p,q}$, then $w^p \in A_p$ and $w^q \in A_q$. We say $V \in B_\infty$ whenever a constant $C > 0$ can be found such that

$$\|V\|_{L^\infty(B)} \lesssim \frac{1}{|B|}\int_B V(x)\,dx$$

holds for any ball B (Li,1999).

Now, we define the generalized weighted Morrey spaces over $\mathbb{R}^n$.

Definition 1. Let $p \in (1,\infty)$ and and let w be a weight defined on $\mathbb{R}^n$. Let ψ be a function taking value on positive real numbers and defined on $\mathbb{R}^n \times \mathbb{R}^+$. The generalized weighted Morrey space, denoted by $\mathcal{M}_\psi^{p,w} = \mathcal{M}_\psi^{p,w}(\mathbb{R}^n)$, is a function space defined as the set collecting any function f on $\mathbb{R}^n$ provided

$$\|f\|_{\mathcal{M}_\psi^{p,w}} = \sup_{a\in\mathbb{R}^n, l>0} \psi(a,l)^{-1} w\big(B(a,l)\big)^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,l))} < \infty.$$

If we take $\psi(a,l) = w\big(B(a,l)\big)^{\frac{1}{q}}$, then $\mathcal{M}_{\psi}^{p,w}$ is the weighted Morrey space (Shirai and Komori, 2009). If w is assumed to be constant almost everywhere, then $\mathcal{M}_{\psi}^{p,w}$ is the generalized weighted Morrey space (Mizuhara , 1993, Nakai, 1991, and Sawano, 2019). Moreover, if $\psi(a,l) = l^{\frac{1}{q}}$ and w is constant almost everywhere, then $\mathcal{M}_{\psi}^{p,w}$ is the classical Morrey spaces (Morrey, 1938).

The example of function in the classical Morrey space is the radial function as in Sawano et al (2020). The example of function in generalized Morrey spaces, weighted Morrey spaces, and generalized weighted Morrey spaces are very dependent on the parameter and the weight function. The reader may see Sawano et al (2019) and Sawano et al (2020).

Results and Discussions

In this part of the paper, we present the main findings of our study along with their discussion, organized into three subsections: (1) main results and some interpretation; (2) the boundedness of Schrödinger operator on generalized weighted Morrey spaces; and (3) proof of the main results.

1. Main results and some interpretations

Given the Schrödinger equations (1) and (2), we obtain the following regularity result for their solution. We begin by observing that the equation admits certain general forms of solutions.

Theorem 1 *Suppose that* $V \in B_\infty$ *and* $0 \le \gamma \le \beta \le 1$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and a weight* w *on* $\mathbb{R}^n$ *such that* $w \in A_{p,q}$. *Suppose the functions* φ_1 *and* φ_2 *satisfy at least one of the following conditions:*

1. *For* $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \frac{dt}{t} \lesssim \varphi_2(x,r).$$
2. *For* $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\sup_{r<t<\infty} \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \lesssim \varphi_2(x,r).$$

If u *is the solution of (1) and* $u \in \mathcal{M}_{\varphi_1}^{p,w^p}$, *then* $\|f\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_2}^{q,w^q}}$.

Theorem 2. *Suppose* $V \in B_\infty$ *and* $0 \le \gamma \le \frac{1}{2} \le 1, \beta - \gamma \ge \frac{1}{2}$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and a weight* w *on* $\mathbb{R}^n$ *such that* $w \in A_{p,q}$. *Suppose the functions* φ_1 *and* φ_2 *satisfy one of the following conditions:*

1. *For* $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \frac{dt}{t} \lesssim \varphi_2(x,r).$$
2. *For* $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\sup_{r<t<\infty} \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \lesssim \varphi_2(x,r).$$

If u *is the solution of (2) and* $u \in \mathcal{M}_{\varphi_1}^{p,w^p}$, *then* $\|f\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_2}^{q,w^q}}$.

In general, we have that $\|f\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}$ with varying values of the parameter in the equations. This implies that if the solution u is in $\mathcal{M}_{\varphi_1}^{p,w^p}$, then the forcing function f must belong to the Morrey spaces $\mathcal{M}_{\varphi_2}^{q,w^q}$. We may extend to the concept of superposition. If $u_1, u_2, \cdots, u_m$ are the solution of (1), then by the linearity of generalized weighted Morrey spaces,

$$\|f\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \left\|\sum_{k=1}^{m} u_k\right\|_{\mathcal{M}_{\varphi_1}^{p,w^p}} \leq \sum_{k=1}^{m} \|u_k\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}.$$

Another interpretation can be obtained by taking the contraposition of the statetment. In the other words, if the forcing function is not in the Morrey spaces $\mathcal{M}_{\varphi_2}^{q,w^q}$, then we cannot have that u in $\mathcal{M}_{\varphi_1}^{p,w^p}$. Thus, if the external forces in equation (1) and (2) fail to satisfy both local regularity and global decay conditions, the solution u also fails to exhibit these properties. This answers the research question. For example, if $f(\cdot) = e^{|\cdot|}$ on $\mathbb{R}^n$ then there is no solution u in (1) and (2) such that the solution is in the Morrey spaces under assumptions of theorems.

We shall prove the theorem in the end of this section after establishing some other theorems which are very useful in proving the main result Theorem 1 and Theorem 2. The other theorems are mostly the boundedness properties of maximal fractional operator and Schrödinger operator.

2. Schrödinger operators on generalized weighted Morrey spaces

Consider the equations (1) and (2). We let the operators L_1 and L_2 where

$$L_1(u) = \left[V^{\gamma}(-\Delta + V)^{-\beta}\right](u)$$

and

$$L_2(u) = \left[V^{\gamma}\nabla(-\Delta + V)^{-\beta}\right](u).$$

The reader may see the works by Akbulut et al (2024), Ambrosio (2022), Dasgupta et al (2024), Dewan (2024), and Fabris et al (2021), for more about Schrödinger operators.

The theorems below establish the connection between the Schrödinger operator (L_1 and L_2) and M_α which motivates the use of the boundedness of M_α as a method to address the question posed in Section 1.

Theorem 3 (Zhong, 1993). *Suppose $V \in B_\infty$ and $0 \leq \gamma \leq \beta \leq 1$. Then,*

$$|L_1 u| \lesssim M_\alpha u, \qquad u \in S_0^\infty$$

where $\alpha = 2(\beta - \gamma)$.

Theorem 4 (Zhong, 1993). *Suppose $V \in B_\infty$ and $0 \leq \gamma \leq \frac{1}{2} \leq 1, \beta - \gamma \geq \frac{1}{2}$. Then,*

$$|L_2 u| \lesssim M_\alpha u, \qquad u \in S_0^\infty$$

where $\alpha = 2(\beta - \gamma) - 1$.

Ramadana and Gunawan (2023) established the following local estimate for M_α as in the theorems:

Theorem 5. *Let $1 < p < \infty$ and a weight w on $\mathbb{R}^n$ such that $w \in A_p$. Then,*

$$\|M_\alpha(u)\|_{L^{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{l(B)}^{\infty} \frac{w^p\big(B(c(B),t)\big)^{\frac{1}{p}}}{w^q\big(B(c(B),t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(c(B),t))} \frac{dt}{t}$$

for $1 < p < \infty$ and $u \in L_{loc}^{p,w^p}$.

Theorem 6. *Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and a weight w on $\mathbb{R}^n$ such that $w \in A_{p,q}$. Suppose the functions φ_1 and φ_2 satisfy*

$$\int_r^\infty \frac{\varphi_1(x,t)w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}}\frac{dt}{t} \lesssim \varphi_2(x,r), \qquad (x,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then, M_α is bounded from $\mathcal{M}_{\varphi_1}^{p,w^p}$ to $\mathcal{M}_{\varphi_2}^{q,w^q}$.

Beside this theorem, we provide the other additional requirement for the boundedness of M_α on generalized weighted Morrey space. This is formalized in the theorem that follows.

Theorem 7. *Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and a weight w on $\mathbb{R}^n$ such that $w \in A_{p,q}$. Suppose the functions φ_1 and φ_2 satisfy*

$$\sup_{r<t<\infty} \frac{\varphi_1(x,t)w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \lesssim \varphi_2(x,r), \qquad (x,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then, M_α is bounded from $\mathcal{M}_{\varphi_1}^{p,w^p}$ to $\mathcal{M}_{\varphi_2}^{q,w^q}$.

Proof. We begin by deriving a local estimate for M_α which resembles the local estimate stated in Theorem 5. Let $u \in \mathcal{M}_{\varphi_1}^{p,w}$, $a \in \mathbb{R}^n$, and $r > 0$. We split u to the form $u = u_1 + u_2$ where $u_1 = u \cdot \mathcal{X}_{B(a,2r)}$. By the boundedness of M_α from L^{p,w^p} to L^{q,w^q},

$$\|M_\alpha(u_1)\|_{L^{q,w^q}(B(a,r))} \le \|M_\alpha(u_1)\|_{L^{q,w^q}} \lesssim \|u_1\|_{L^{p,w^p}} = \|u\|_{L^{p,w^p}(B(a,2r))}$$

By using Lemma 2.12 in (Ramadana & Gunawan, 2023), w obtain that

$$\|M_\alpha(u_1)\|_{L^{q,w^q}(B(a,r))} \lesssim \|u\|_{L^{p,w^p}(B(a,2r))} \lesssim w^q(B)^{\frac{1}{q}} \sup_{r<t<\infty} \frac{w^p\big(B(a,t)\big)^{\frac{1}{p}}}{w^q(Ba,t)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))}.$$

Next, we estimate the norm of u_2 over $B(a,r)$ under L^{q,w^q} and WL^{q,w^q}. Let $x \in B(a,r)$ and $t > 0$. If $y \in B(a,r)$ and $y \in B(a,2r)^c$, then

$$r = 2r - r \le |y-a| - |a-x| \le |y-x| < t$$

which actually means that

$$\int_{B(x,t)\cap B(a,2r)^c} |u(y)|\, dy = 0$$

for $t \le r$. Moreover, $B(x,t) \cap B(a,2r)^c \subseteq B(a,2t)$ for $r < t$. In fact, we may see that

$$|y-a| \le |y-x| + |x-a| \le t + r < 2t.$$

Hence, the estimate below holds by Hölder's inequality and $A_{p,q}$ condition.

$$\begin{aligned}
M_\alpha(u_2)(x) &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2(y)|dy \\
&= \max\left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2(y)|dy,\ \sup_{0<t\le r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2(y)|dy \right) \\
&= \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2(y)|dy \\
&= \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)\cap B(a,2r)^c} |u(y)|dy \\
&\le \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(a,2r)} |u(y)|dy \\
&\cong \sup_{t>2r} \frac{1}{|B(a,t)|^{1-\frac{\alpha}{n}}} \int_{B(a,r)} |u(y)|dy \\
&\le \sup_{t>2r} \frac{1}{|B(a,t)|^{1-\frac{\alpha}{n}}} \|u\|_{L^{p,w^p}(B(a,t))} \|w^{-p}\|_{L^{p'}(B(a,t))} \\
&\lesssim \sup_{t>r} \frac{1}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))}.
\end{aligned}$$

Hence, by taking the norm of L^{q,w^q} on $M_\alpha(u_2)$ over $B(a,r)$ we obtain

$$\|M_\alpha(u_2)\|_{L^{q,w^q}B(a,r)} \lesssim w^q\big(B(a,r)\big)^{\frac{1}{q}} \sup_{t>2r} \frac{1}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))}.$$

By combining the estimates both for $M_\alpha(u_1)$ and $M_\alpha(u_2)$ and using the linearity of M_α on weighted Lebesgue spaces, we obtain that

$$\|M_\alpha(u)\|_{L^{q,w^q}B(a,r)} \lesssim w^q\big(B(a,r)\big)^{\frac{1}{q}} \sup_{t>2r} \frac{1}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))}.$$

Hence, by the definition of the norm of $\mathcal{M}_{\varphi_1}^{p,w^p}$ and $\mathcal{M}_{\varphi_2}^{q,w^q}$,

$$\begin{aligned}
\|M_\alpha(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(a,r)} \frac{1}{w^q\big(B(a,r)\big)^{\frac{1}{q}}} \|M_\alpha(u)\|_{L^{q,w^q}B(a,r)} \\
&\lesssim \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(a,r)} \frac{1}{w^q\big(B(a,r)\big)^{\frac{1}{q}}} w^q\big(B(a,r)\big)^{\frac{1}{q}} \sup_{t>r} \frac{1}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(a,r)} \sup_{t>r} \frac{1}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(a,r)} \sup_{t>r} \frac{\varphi_1(a,r)}{\varphi_1(a,r)} \frac{1}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \|u\|_{L^{p,w^p}(B(a,t))} \\
&\leq \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(a,r)} \sup_{t>r} \varphi_1(a,t) \frac{w^p\big(B(a,t)\big)^{\frac{1}{p}}}{w^q\big(B(a,t)\big)^{\frac{1}{q}}} \\
&\lesssim \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}.
\end{aligned}$$

This proves that M_α is bounded from $\mathcal{M}_{\varphi_1}^{p,w^p}$ to $\mathcal{M}_{\varphi_2}^{q,w^q}$ ■

From these results, we derive the following the other results illustrating the boundedness properties of the Schrödinger operator acting on generalized weighted Morrey spaces.

Theorem 8. *Suppose* $V \in B_\infty$ *and* $0 \leq \gamma \leq \beta \leq 1$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \leq p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and a weight* w *on* $\mathbb{R}^n$ *such that* $w \in A_{p,q}$. *Suppose that* φ_1 *and* φ_2 *are two functions on* $\mathbb{R}^n \times \mathbb{R}^+$ *such that one of the following conditions bolds:*

3. *For* $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \frac{dt}{t} \lesssim \varphi_2(x,r).$$
4. *For* $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\mathop{sup}_{r<t<\infty} \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \lesssim \varphi_2(x,r).$$

Then, L_1 *is bounded from* $\mathcal{M}_{\varphi_1}^{p,w^p}$ *to* $\mathcal{M}_{\varphi_2}^{q,w^q}$.

Proof. By using Theorem 3 and Theorem 6, and by combining those with Theorem 7 we obtain the following inequalities:

$$\|L_1(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|M_\alpha(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}$$

which allow us to conclude the boundedness of L_1 from $\mathcal{M}_{\varphi_1}^{p,w^p}$ to $\mathcal{M}_{\varphi_2}^{q,w^q}$ ■

Theorem 9. *Suppose $V \in B_\infty$ and $0 \le \gamma \le \frac{1}{2} \le 1, \beta - \gamma \ge \frac{1}{2}$. Let $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and a weight w on $\mathbb{R}^n$ such that $w \in A_{p,q}$. Suppose φ_1 and φ_2 are functions satisfying one of the following conditions:*

1. *For $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,*
$$\int_r^\infty \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \frac{dt}{t} \lesssim \varphi_2(x,r).$$
2. *For $(x,r) \in \mathbb{R}^n \times \mathbb{R}^+$,*
$$\sup_{r<t<\infty} \frac{\varphi_1(x,t) w^p\big(B(x,t)\big)^{\frac{1}{p}}}{w^q\big(B(x,t)\big)^{\frac{1}{q}}} \lesssim \varphi_2(x,r).$$

Then, L_2 is bounded from $\mathcal{M}_{\varphi_1}^{p,w^p}$ to $\mathcal{M}_{\varphi_2}^{q,w^q}$.

Proof. Observing Theorems 4, 6, and 7, we see that the following inequalities hold.
$$\|L_2(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|M_\alpha(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}$$
Hence, we obtain the boundedness of L_2 from $\mathcal{M}_{\varphi_1}^{p,w^p}$ to $\mathcal{M}_{\varphi_2}^{q,w^q}$ ■

3. Proof of the Main Results

We can now proceed to prove the main result.

Proof of Theorem 1. Suppose the equation (1) and $u \in \mathcal{M}_{\varphi_1}^{p,w^p}$ is the solution of the Schrödinger equation (1). We make use of Theorem 8 to obtain
$$\|f\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} = \|L_1(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}$$
that proves the theorem ■

Proof of Theorem 2. Suppose the equation (2) and $u \in \mathcal{M}_{\varphi_1}^{p,w^p}$ is the solution of the Schrödinger equation (2). Then, Theorem 8 implies
$$\|f\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} = \|L_2(u)\|_{\mathcal{M}_{\varphi_2}^{q,w^q}} \lesssim \|u\|_{\mathcal{M}_{\varphi_1}^{p,w^p}}$$
as desired ■

Conclusion

Some conditions are obtained in order to ensure the boundedness of fractional maximal operator and imply the boundedness properties of Schrödinger operator on generalized weighted Morrey spaces. By using the boundedness properties, it is obtained some regularity properties of the solution of Schrödinger equation with forcing function. When the forcing function does not maintain local regularity and global decay, the solution of the equation similarly lacks these properties. This finding helps to better understand the dynamics of quantum particles in quantum systems.

Acknowledgments

The author would like to thank all parties who have been willing to engage in discussions regarding Morrey spaces and the Schrödinger equation.

Conflicts of interest

The author confirms the absence of any conflicts of interest.

Declaration of AI use

The author declares that no artificial intelligence (AI) tools were used in the conduct of the research. All mathematical results and proofs were produced entirely by the author.

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