Journal of Natural Sciences and Mathematics Research
J. Nat. Scien. & Math. Res. Vol. 11 No. 2 (2025) 136-145
ISSN: 2460-4453 (Online); 2614-6487 (Printed)
Available online at http://journal.walisongo.ac.id/index.php/jnsmr

Efficient Numerical Method for Generating Closed Form Solution for Nonlinear Bratu Differential Equations

Otaide Ikechukwu Jackson[1*], and Ekuma-Okereke Enyinnaya[1]
[1] Department of Mathematics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria,

* Corresponding author's e-mail: otaideikechukwu@gmail.com

ABSTRACT
This study presents a numerical solution of the Bratu differential equations (BDE) using the Sumudu transform series decomposition technique (STSDT). The process combines the Adomian polynomials (AP), series expansion (SE), and Sumudu transform (ST), and it ultimately converges perfectly to the exact solution. Examining four test problems demonstrates that the strategy converges more effectively than the literature-based approach. Calculations were performed using Maple 2022 software.

Keywords:
Sumudu transform, Adomian decomposition method, Convergence analysis, Bratu differential equations, Approximate solution.

Introduction

Consider the nonlinear Bratu differential equation of the form:

$$\frac{d^2W}{d\xi^2}+\gamma e^{\alpha W}=0;\quad 0<\xi<1;\quad \gamma,\alpha\ \in \Re \tag{1}$$

with initial conditions $W(0)=W'(0)=0\,.$ (2)

The significance of Equation (1), a one-dimensional Bratu-type problem, stems from the first thermal combustion theory that was established by simplifying the solid fuel ignition model. The development of nonlinear differential equations has been crucial to numerous disciplines, including chemical kinetics, fluid mechanics, and plasma physics. Because of this, scientists are interested to find the exact solution to nonlinear differential equations. Many applications of the Bratu equation can be found in the fields of thermal reaction, radiative heat transfer, chemical reactor theory and fuel ignition model of thermal combustion. Engineers have recently discovered applications for the Bratu equation, such as the electrospinning method used to create nanofibers. The Boundary Value Problem (BVP) first appeared in the modeling of electrically conducting materials, and nonlinear elliptic equations with such boundary conditions have been utilized to explain thermal explosions in the field of combustion theory. As a benchmark order to verify the precision and effectiveness of numerical methods, the Bratu equation is frequently employed. Consequently, a great deal of research on Bratu-type differential equations has been done.

For instance, Salem & Thanoon, (2022) used a perturbation method for the solution of the Bratu's type equation. Similarly, Zarebnia & Hoshyar (2014) used spline method for the solution of Bratu's type equation. While Wazwaz (2016) used method of successive differentiation for solving Bratu's type equations. The solution of Bratu equation was verified by Saravi et al. (2013) using the He's variational Iteration Method. Numerical solution Bratu equations was obtained by Fenta & Derese (2019) using sixth order Runge-Kutta seven stages method. A New Improved Variational Homotopy Perturbation Method was used to address Bratu-Type Problems by (Ezekiel, 2013). Furthermore, (Hassan & Semary, 2013) used Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method. Numerical solution of Bratu problem using the method of weighted residual was implemented by (Aregbesola, 2003). Adomians decomposition method for the treatment of Bratu's type equations was handled by (Wazwaz, 2005).

Finally, (Changqing & Jianhua, 2013) used Chebyshev wavelets method for the solution of Bratu's problem. While these methods are effective, a significant gap exists in the development of a technique that seamlessly combines the computational efficiency of integral transforms with the simplicity and elegance of series decomposition, thereby avoiding the complexities of the aforementioned approaches. Specifically, the Sumudu Transform has been underutilized for this class of problems. Although its potential has not been completely utilized in conjunction with a simple series decomposition method for the nonlinear Bratu equation with the specified initial conditions. The numerical solution of Bratu's nonlinear differential equation (1) with the provided initial conditions (2) is produced in this study using the Sumudu Transform Series Decomposition echnique (STSDT). The suggested approach functions effectively, and the preliminary findings are reassuring and trustworthy. Ultimately, the solution is provided in an infinite series that converges perfectly to the exact solution.

Preliminaries of Sumudu Transform

1 Definition (Sudhanshu et al., 2020; Aggarwal & Sharma, 2019; Akinola et al., 2016)**.** The Sumudu transform of a function $W(\xi)$ is defined by

$$\mathcal{S}[W(\xi)] = w(s) = \frac{1}{s}\int_0^\infty W(\xi)e^{-\frac{\xi}{s}}d\xi$$

where the Sumudu transform operator is represented by $\mathcal{S}$.

Thus, $W(\xi)$ is called the inverse Sumudu transform of $w(s)$ and this is written as

$$W(\xi) = \mathcal{S}^{-1}[w(s)]$$

2 Properties of Sumudu transform (Sudhanshu et al., 2020):

$$\mathcal{S}\{aW_1(\xi) + bW_2(\xi)\} = a\mathcal{S}\{W_1(\xi)\} + b\mathcal{S}\{W_2(\xi)\} \quad (3)$$

$$\mathcal{S}\{W(a\xi)\} = w(as) \quad (4)$$

$$\mathcal{S}\left\{e^{a\xi}W(\xi)\right\} = \left(\frac{1}{1-as}\right)w\left(\frac{s}{1-as}\right) \quad (5)$$

$$\mathcal{S}\{W'(\xi)\} = \frac{1}{s}w(s) - \frac{1}{s}W(0) \quad (6)$$

$$\mathcal{S}\{W''(\xi)\} = \frac{1}{s^2}w(s) - \frac{1}{s^2}W(0) - \frac{1}{s}W'(0) \quad (7)$$

$$\mathcal{S}\left\{W^{(n)}(\xi)\right\} = \frac{1}{s^n}w(s) - \frac{1}{s^n}W(0) - \frac{1}{s^{n-1}}W'(0) - \cdots - \frac{1}{s}W^{(n-1)}(0) \quad (8)$$

$$\mathcal{S}\{W_1(\xi) * W_2(\xi)\} = s\mathcal{S}\{W_1(\xi)\}\mathcal{S}\{W_2(\xi)\} \quad (9)$$

Sumudu Transform Series Decomposition Technique (STSDT) (Akinola et al., 2016)

Examine the following general nonlinear, nonhomogeneous differential equation:

$$LW(\xi) + RW(\xi) + NW(\xi) = g(\xi) \quad (10)$$

Where L stands for the linear differential operator of highest order, the linear differential operator of order less than L is denoted by R, a series expansion is assumed for the source term g and the non-linear differential operator N.

On both sides of (10), the Sumudu transform is applied to yield:

$$\mathcal{S}[LW(\xi)] + \mathcal{S}[RW(\xi)] + \mathcal{S}[NW(\xi)] = \mathcal{S}[g(\xi)] \quad (11)$$

By utilizing the Sumudu transform's differentiation property (8) in (11), we can achieve

$$\frac{\mathcal{S}[W(\xi)]}{w^m} - \sum_{k=0}^{m-1} \frac{W(\xi)^{(k)}(0)}{w^{(m-k)}} + \mathcal{S}[RW(\xi)] + \mathcal{S}[NW(\xi)] = \mathcal{S}[g(\xi)] \quad (12)$$

Where

$$\sum_{k=0}^{m-1} \frac{W(\xi)^{(k)}(0)}{w^{(m-k)}} = \sum_{k=0}^{m-1} \frac{W(0)^{(k)}}{w^{(m-k)}}$$

Simplifying (12) further we have that

$$\mathcal{S}[W(\xi)] - \sum_{k=0}^{m-1} \frac{W(\xi)^{(k)}(0)}{w^{(m-k)}} + w^m(\mathcal{S}[RW(\xi)] + \mathcal{S}[NW(\xi)] - \mathcal{S}[g(\xi)]) = 0 \quad (13)$$

Applying the inverse Sumudu transform on (13) yields

$$W(\xi) = G(\xi) - \mathcal{S}^{-1}\{w^m(\mathcal{S}[RW(\xi)] + \mathcal{S}[NW(\xi)])\} \quad (14)$$

with $G(\xi) = \mathcal{S}^{-1}\left[w^m\left(\sum_{k=0}^{m-1} \frac{W(\xi)^{(k)}(0)}{w^{(m-k)}} + \mathcal{S}[g(\xi)]\right)\right]$ (15)

Where the term resulting from the source term and the specified initial conditions is indicated by $G(\xi)$.

As an infinite series, (14) has the following solution:

$$W(\xi) = \sum_{n=0}^{\infty} W_n(\xi) \quad (16)$$

Decomposing the nonlinear term yields:

$$NW(\xi) = \sum_{n=0}^{\infty} A_n(w_0, w_1, \dots, w_n) \quad (17)$$

where A_n denotes the Adomian polynomials, which can be calculated by applying the technique described in [20]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i w_i\right)\right]_{\lambda=0}, n = 0,1,2,3, \dots \quad (18)$$

Substituting (16) and (17) into (14) gives

$$\sum_{n=0}^{\infty} W_{n+1}(\xi) = G(\xi) - \mathcal{S}^{-1}\{\mathcal{S}w^m([R\sum_{n=0}^{\infty} W_n(\xi)] + [\sum_{n=0}^{\infty} A_n])\} \quad (19)$$

The series solution can be obtained by continuously simplifying (19).

$$\left.\begin{aligned} W_0(\xi) = G(\xi) = \mathcal{S}^{-1}\left[w^m\left(\sum_{k=0}^{m-1} \frac{W(\xi)^{(k)}(0)}{w^{(m-k)}} + \mathcal{S}[g(\xi)]\right)\right] \\ W_{n+1}(\xi) = -\mathcal{S}^{-1}\{\mathcal{S}w^m([RW_n(\xi)] + [A_n])\}: n \geq 0 \end{aligned}\right\} \quad (20)$$

Thus, the series solution is ascertained by applying the recursive relation (20).

$$W(\xi) = \sum_{n=0}^{\infty} W_n(\xi) = W_0(\xi) + W_1(\xi) + W_2(\xi) + W_3(\xi) + W_4(\xi) + \cdots \quad (21)$$

Convergence of the method

A complete normed linear space addressing the convergence of the Sumudu Transform Series Decomposition Technique (STSDT) is provided in this section. A sequence of function recurrences is produced by applying the approach to the nonlinear differential equation. It is presumed that the solution to the given problem is located at the limit of this sequence.

Theorem 4.1
Let $X = (C[0,T], \|.\|)$ be a complete normed linear space of all continuous functions endowed with the maximum norm where $t \longmapsto W(t)$. Set $W_{n+1}(\xi) = W_0(\xi) - \mathfrak{S}^{-1}(t)\left[\mathfrak{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\right], n \geq 0$, where $N[W_n(\xi)] = \sum_{n=0}^{\infty} A_n(W_0, W_1, \dots, W_n)$. Define a mapping $T: X \longrightarrow X$ as follows $T(\xi) \coloneqq W_0(\xi) - \mathfrak{S}^{-1}(t)\left[\mathfrak{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\right], n \geq 0$ and suppose R and N are Lipschitzian mappings such that there exists two Lipschitzian constants $k_1, k_2 < 1$ and s^m with $0 < k < 1$, where $k = (k_1 + k_2)s^m$, then the recursive relation (20) has a unique solution.
Proof.
There is at most one solution. Assume that W^* and W as exact and approximate solutions respectively such that $W^*(\xi) \neq W(\xi)$. Since

$$\|RW^* - RW\| \leq k_1\|W^* - W\|, \forall\, W^*, W \in X \tag{22}$$

$$\|NW^* - NW\| \leq k_2\|W^* - W\|, \forall\, W^*, W \in X \tag{23}$$

for all W^* and W as exact and approximate solutions respectively. We want to show the uniqueness of T. To do this, we use the following estimate and the properties mentioned above to get:

$$\begin{aligned}
\|TW^* - TW\| &= \Big|W_0(\xi) - \mathcal{S}^{-1}(t)\Big[\mathcal{S}(t)s^m\big[R(W^*{}_n(\xi)) + N(W^*{}_n(\xi))\big]\Big] \\
&\qquad - \Big[W_0(\xi) - \mathcal{S}^{-1}(t)\big[\mathcal{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\big]\Big]\Big| \\
&= \Big|-\mathcal{S}^{-1}(t)\Big[\mathcal{S}(t)s^m\big[R(W^*{}_n(\xi)) + N(W^*{}_n(\xi))\big]\Big] + \mathcal{S}^{-1}(t)\big[\mathcal{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\big]\Big| \\
&= \Big|\mathcal{S}^{-1}(t)\big[\mathcal{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\big] - \mathcal{S}^{-1}(t)\Big[\mathcal{S}(t)s^m\big[R(W^*{}_n(\xi)) + N(W^*{}_n(\xi))\big]\Big]\Big| \\
&\leq \max_{t\in[0,T]} \Big|\mathcal{S}^{-1}(t)\big[\mathcal{S}(t)s^m[R(W_n(\xi)) - R(W^*{}_n(\xi))]\big] + \mathcal{S}^{-1}(t)\big[\mathcal{S}(t)s^m[N(W_n(\xi)) - N(W^*{}_n(\xi))]\big]\Big| \\
&= \max_{t\in[0,T]} |\mathcal{S}^{-1}(t)[\mathcal{S}(t)s^m]|\|R(W_n(\xi)) - R(W^*{}_n(\xi))\| + |\mathcal{S}^{-1}(t)[\mathcal{S}(t)s^m]|\|N(W_n(\xi)) - N(W^*{}_n(\xi))\| \\
&\leq \max_{t\in[0,T]} (k_1 + k_2)|\mathcal{S}^{-1}(t)[\mathcal{S}(t)s^m]|\|W_n(\xi) - W^*{}_n(\xi)\| \\
&\leq \max_{t\in[0,T]} (k_1 + k_2)s^m\|W_n(\xi) - W^*{}_n(\xi)\| \\
&\leq \max_{t\in[0,T]} (k_1 + k_2)s^m\|W^*(\xi) - W(\xi)\|, \text{ since } 0 < k < 1, \text{ we get the contradiction}
\end{aligned}$$

$\|W^*(\xi) - W(\xi)\| = \|W(\xi) - W^*(\xi)\| \Rightarrow W(\xi) = W^*(\xi)$. This completes the proof. ■

Theorem 4.2
Let $X = (C[0,T], \|.\|)$ be a complete normed linear space of all continuous functions endowed with the maximum norm with $t \longmapsto W(t)$. Set $W_{n+1}(\xi) = W_0(\xi) - \mathfrak{S}^{-1}(t)\left[\mathfrak{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\right], n \geq 0$, where $N[W_n(\xi)] = \sum_{n=0}^{\infty} A_n(W_0, W_1, \dots, W_n)$. Define a mapping $T: X \longrightarrow X$ as follows $T(\xi) \coloneqq W_0(\xi) - \mathfrak{S}^{-1}(t)\left[\mathfrak{S}(t)s^m[R(W_n(\xi)) + N(W_n(\xi))]\right], n \geq 0$ and suppose R and N are Lipschitzian mappings such that there exists two Lipschitzian constants $k_1, k_2 < 1$ and s^m with $0 < k < 1$, where $k = (k_1 + k_2)s^m$, then the recursive relation (20) is convergent.
Proof.

To prove that recursive relation (20) is convergent, we will show that for any given point $m^* \in X$, the n^{th} partial sum of the series $\sum_{k=0}^{n} W_k(\xi)$ is Cauchy. To see this, let $M_n = \sum_{k=0}^{n} W_k(\xi)$ be the n^{th} partial sum of the series in (20).
Now,

$$\|M_n - M_m\| = \max_{t\in[0,T]} |M_n - M_m| = \max_{t\in[0,T]} |\textstyle\sum_{k=m+1}^{n} W_k(\xi)|$$

$$\|M_n - M_m\| = \max_{t\in[0,T]} \left|\textstyle\sum_{k=m+1}^{n}(W_0(\xi) - \mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[R(W_{k-1}(\xi)) + N(W_{k-1}(\xi))]\right]\right|$$

$$\le \max_{t\in[0,T]} \left|\mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[R(\textstyle\sum_{k=m+1}^{n} W_{k-1}(\xi))]\right] + \mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[N(\textstyle\sum_{k=m+1}^{n} W_{k-1}(\xi))]\right]\right|$$

$$\le \max_{t\in[0,T]} \left|\mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[\textstyle\sum_{k=m+1}^{n} R(M_{n-1}) - R(M_{m-1})]\right] + \mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[\textstyle\sum_{k=m+1}^{n} N(M_{n-1}) - N(M_{m-1})]\right]\right|$$

$$\le \max_{t\in[0,T]} \left|\mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[R(M_{n-1}) - R(M_{m-1})]\right] + \mathcal{S}^{-1}(t)\left[\mathcal{S}(t)s^m[N(M_{n-1}) - N(M_{m-1})]\right]\right|$$

$$\le \max_{t\in[0,T]} (k_1 + k_2)[\mathcal{S}^{-1}(t)[\mathcal{S}(t)s^m\|M_{n-1} - M_{m-1}\|]]$$

$$\le (k_1 + k_2)s^m\|M_{n-1} - M_{m-1}\|$$

$$= k\|M_{n-1} - M_{m-1}\|.$$

Let $n = m + 1$.By Induction, we have:
$\|M_{m+1} - M_m\| = (k)^m\|M_1 - M_0\|$ and this is a contraction.
Since $0 < k < 1$, it results that $(k)^m \to 0$ as $m \to \infty$.
In the sequel,

$$\|M_{m+1} - M_m\| = |k|^m\|M_1 - M_0\|$$

Showing that $\{M_m\}_{m=0}^{\infty}$ is a Cauchy sequence.
Since $X = (C[O,T], \|.\|)$ is a complete normed linear space, then $\{M_m\}_{m=0}^{\infty}$ converges to some $m^* \in X$ ■

Numerical Application

This section presents the solution of four Bratu-type problems using the proposed methodology. The outcomes demonstrate the suggested scheme's dependability and effectiveness.

Example 5.1 (Salem & Thanoon, 2022): Consider the Bratu-type problem (1), by choosing $\gamma = -2$ and $\alpha = 1$ to obtain:

$$\frac{d^2W}{d\xi^2} - 2e^W = 0, \quad 0 < \xi < 1, \tag{24}$$

$$W(0) = W'(0) = 0 \tag{25}$$

The exact solution of the system (24) is given in (Salem & Thanoon, 2022) as:

$$W^*(\xi) = \xi^2 + \frac{\xi^4}{6} + \frac{2\xi^6}{45} + \frac{17\xi^8}{1260} + \cdots$$

Solution.
Applying the Sumudu transform on both sides of (24) gives:

$$\mathcal{S}\left(\frac{d^2W}{d\xi^2} - 2e^W\right) = \mathcal{S}(0)$$

Using (7), (25), (17) and (20) gives the following:

$$W_{n+1} = 2\mathcal{S}^{-1}[s^2\mathcal{S}(A_n)] : n \ge 0$$

For $n = 0$

$$W_1 = 2\mathcal{S}^{-1}[s^2\mathcal{S}(A_0)] = 2\mathcal{S}^{-1}[s^2\mathcal{S}(e^{W_0})] = \xi^2 \ (W_0 = 0)$$

For $n = 1$

$$W_2 = 2\mathcal{S}^{-1}[s^2\mathcal{S}(A_1)] = 2\mathcal{S}^{-1}[s^2\mathcal{S}(W_1 e^{W_0})] = \frac{\xi^4}{6}$$

For $n = 2$

$$W_3 = 2\mathcal{S}^{-1}[s^2\mathcal{S}(A_2)] = 2\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{W_0}\left[W_2 + \frac{1}{2!}{W_1}^2\right]\right)\right] = \frac{2\xi^6}{45}$$

For $n = 3$

$$W_4 = 2\mathcal{S}^{-1}[s^2\mathcal{S}(A_3)] = 2\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{W_0}\left[W_3 + W_1W_2 + \frac{1}{3!}{W_1}^3\right]\right)\right] = \frac{17\xi^8}{1260}$$

Hence the approximate solution is given as
$W(\xi) = \xi^2 + \frac{\xi^4}{6} + \frac{2\xi^6}{45} + \frac{17\xi^8}{1260} + \cdots$ which is the same as the exact solution.

Example 5.2 (Salem & Thanoon, 2022): Consider the Bratu-type problem (1), by choosing $\gamma = -1$ and $\alpha = 2$ to obtain:

$$\frac{d^2W}{d\xi^2} - e^{2W} = 0, \quad 0 < \xi < 1, \tag{26}$$

$$W(0) = W'(0) = 0 \tag{27}$$

The exact solution of the system (26) is given in (Salem & Thanoon, 2022) as:

$$W^*(\xi) = \frac{\xi^2}{2} + \frac{\xi^4}{12} + \frac{\xi^6}{45} + \frac{17\xi^8}{2520} + \cdots$$

Solution.
Applying the Sumudu transform on both sides of (26) gives:
$\mathcal{S}\left(\frac{d^2W}{d\xi^2} - e^{2W}\right) = \mathcal{S}(0)$
Using (7), (27), (17) and (20) gives the following:
$W_{n+1} = \mathcal{S}^{-1}[s^2\mathcal{S}(A_n)] : n \geq 0$
For $n = 0$

$$W_1 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_0)] = \mathcal{S}^{-1}[s^2\mathcal{S}(e^{2W_0})] = \frac{\xi^2}{2} \ \ (W_0 = 0)$$

For $n = 1$

$$W_2 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_1)] = \mathcal{S}^{-1}[s^2\mathcal{S}(2W_1 e^{2W_0})] = \frac{\xi^4}{12}$$

For $n = 2$

$$W_3 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_2)] = 2\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{2W_0}\left[2W_2 + \frac{1}{2!}4{W_1}^2\right]\right)\right] = \frac{\xi^6}{45}$$

For $n = 3$

$$W_4 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_3)] = \mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{W_0}\left[2W_3 + 4W_1W_2 + \frac{1}{3!}8{W_1}^3\right]\right)\right] = \frac{17\xi^8}{2520}$$

Hence the approximate solution is given as
$W(\xi) = \frac{\xi^2}{2} + \frac{\xi^4}{12} + \frac{\xi^6}{45} + \frac{17\xi^8}{2520} + \cdots$ which is the same as the exact solution.

Example 5.3: Consider the Bratu-type problem (1), by choosing $\gamma = \alpha = -1$ to obtain:

$$\frac{d^2W}{d\xi^2} - e^{-W} = 0, \quad 0 < \xi < 1, \qquad (28)$$

$$W(0) = W'(0) = 0 \qquad (29)$$

The exact solution of the system (28) is given as:

$$W^*(\xi) = \frac{\xi^2}{2} - \frac{\xi^4}{24} + \frac{\xi^6}{180} - \frac{17\xi^8}{20160} + \cdots$$

Solution.
Applying the Sumudu transform on both sides of (28) gives:
$\mathcal{S}\left(\frac{d^2W}{d\xi^2} - e^{-W}\right) = \mathcal{S}(0)$
Using (7), (29), (17) and (20) gives the following:
$W_{n+1} = \mathcal{S}^{-1}[s^2\mathfrak{C}(A_n)] : n \geq 0$
For $n = 0$

$$W_1 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_0)] = \mathcal{S}^{-1}[s^2\mathcal{S}(e^{-W_0})] = \frac{\xi^2}{2} \quad (W_0 = 0)$$

For $n = 1$

$$W_2 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_1)] = \mathcal{S}^{-1}[s^2\mathcal{S}(-W_1e^{-W_0})] = \frac{-\xi^4}{24}$$

For $n = 2$

$$W_3 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_2)] = \mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{-W_0}\left[-W_2 + \frac{1}{2!}{W_1}^2\right]\right)\right] = \frac{\xi^6}{180}$$

For $n = 3$

$$W_4 = \mathcal{S}^{-1}[s^2\mathcal{S}(A_3)] = \mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{-W_0}\left[-W_3 + W_1W_2 - \frac{1}{3!}{W_1}^3\right]\right)\right] = \frac{-17\xi^8}{20160}$$

Hence the approximate solution is given as

$W(\xi) = \frac{\xi^2}{2} - \frac{\xi^4}{24} + \frac{\xi^6}{180} - \frac{17\xi^8}{20160} + \cdots$ which is the same as the exact solution.

Example 5.4: Consider the Bratu-type problem (1), by choosing $\gamma = -\frac{2}{3}$ and $\alpha = \frac{1}{2}$ to obtain:

$$\frac{d^2W}{d\xi^2} - \frac{2}{3}e^{\frac{1}{2}W} = 0, \quad 0 < \xi < 1, \qquad (30)$$

$$W(0) = W'(0) = 0 \qquad (31)$$

The exact solution of the system (30) is given as:

$$W^*(\xi) = \frac{\xi^2}{3} + \frac{\xi^4}{108} + \frac{\xi^6}{2430} + \frac{17\xi^8}{816480} + \cdots$$

Solution.
Applying the Sumudu transform on both sides of (30) gives:
$\mathcal{S}\left(\frac{d^2W}{d\xi^2}-\frac{2}{3}e^{\frac{1}{2}W}\right)=\mathcal{S}(0)$
Using (7), (31), (17) and (20) gives the following:
$W_{n+1}=\frac{2}{3}\mathcal{S}^{-1}[s^2\mathcal{S}(A_n)]:n\geq 0$
For $n=0$

$$W_1=\frac{2}{3}\mathcal{S}^{-1}[s^2\mathcal{S}(A_0)]=\frac{2}{3}\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{\frac{1}{2}W_0}\right)\right]=\frac{\xi^2}{3}\ \ (W_0=0)$$

For $n=1$

$$W_2=\frac{2}{3}\mathcal{S}^{-1}[s^2\mathcal{S}(A_1)]=\frac{2}{3}\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(\frac{1}{2}W_1e^{\frac{1}{2}W_0}\right)\right]=\frac{\xi^4}{108}$$

For $n=2$

$$W_3=\frac{2}{3}\mathcal{S}^{-1}[s^2\mathcal{S}(A_2)]=\frac{2}{3}\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{\frac{1}{2}W_0}\left[\frac{1}{2}W_2+\frac{1}{8}{W_1}^2\right]\right)\right]=\frac{\xi^6}{2430}$$

For $n=3$

$$W_4=\frac{2}{3}\mathcal{S}^{-1}[s^2\mathcal{S}(A_3)]=\frac{2}{3}\mathcal{S}^{-1}\left[s^2\mathcal{S}\left(e^{\frac{1}{2}W_0}\left[\frac{1}{2}W_3+\frac{1}{4}W_1W_2+\frac{1}{48}{W_1}^3\right]\right)\right]=\frac{17\xi^8}{816480}$$

Hence the approximate solution is given as
$W(\xi)=\frac{\xi^2}{3}+\frac{\xi^4}{108}+\frac{\xi^6}{2430}+\frac{17\xi^8}{816480}+\cdots$ which is the same as the exact solution.

Results and Discussion

Table 1. Comparison of the absolute errors for example 5.1

ξ	Exact solution	Approximate solution	Absolute Error by the proposed method	Absolute error by Perturbation Method (Salem & Thanoon, 2022)
0.1	0.01001671124	0.01001671124	0.00000000e-00	1.117×10^{-8}
0.2	0.04026954565	0.04026954565	0.00000000e-00	7.2635×10^{-7}
0.3	0.09138328521	0.09138328521	0.00000000e-00	0.00000849054
0.4	0.1644575533	0.1644575533	0.00000000e-00	0.0000494120
0.5	0.2611638145	0.2611638145	0.00000000e-00	0.0001968626
0.6	0.3839002149	0.3839002149	0.00000000e-00	0.0006183771
0.7	0.5360233017	0.5360233017	0.00000000e-00	0.0016503540
0.8	0.7221811037	0.7221811037	0.00000000e-00	0.0039113549
0.9	0.9487774909	0.9487774909	0.00000000e-00	0.0084672048

Discussion

In summary, the findings clearly show that the hybrid STSDT is a powerful, precise, and effective approach. It successfully addresses a gap in the literature by integrating the advantages of the Sumudu Transform, Adomian decomposition, and series expansion into a coherent and reliable framework. The STSDT is a highly valuable addition to the toolkit of methods for solving nonlinear initial value problems in applied mathematics and engineering because of its performance, which is validated by zero absolute error and superior to existing approaches in literature, as well as its strong theoretical foundation.

Table 2. Comparison of the absolute errors for example 5.2

ξ	**Exact solution**	**Approximate solution**	**Absolute Error by the proposed method**	**Absolute error by Perturbation Method (Salem & Thanoon, 2022)**
0.1	0.00500835562	0.00500835562	0.00000000e-00	2.3386×10^{-8}
0.2	0.02013477282	0.02013477282	0.00000000e-00	0.00000149142
0.3	0.04569164261	0.04569164261	0.00000000e-00	0.00001688860
0.4	0.08222877663	0.08222877663	0.00000000e-00	0.00009410781
0.5	0.1305819072	0.1305819072	0.00000000e-00	0.0003551506
0.6	0.1919501074	0.1919501074	0.00000000e-00	0.0010464518
0.7	0.2680116508	0.2680116508	0.00000000e-00	0.0025969723
0.8	0.3610905518	0.3610905518	0.00000000e-00	0.0056791045
0.9	0.4743887455	0.4743887455	0.00000000e-00	0.0112663280

Conclusion

The efficiency of the Sumudu transform series decomposition technique (STSDT) in generating approximations for nonlinear Bratu differential equation solutions has been shown in this study. The method incorporates the Sumudu transform, series expansion, and Adomian polynomials. The method offers solutions for series that arise in real-world scenarios and swiftly converge. The suggested strategy performs better than methods outlined in the literature, as seen in Tables 1 and 2. The outcomes additionally indicated the effectiveness and utility of the proposed technique as a tool for addressing the problem class under investigation.

Acknowledgements

Special thanks to Federal University of Petroleum Resources Effurun, Delta State, Nigeria for creating a conducive environment for research. Also, providing adequate facilities to promote research work.

Competing Interests

The authors do not have any known competing interests

References

Aggarwal, S., & Sharma, S. D. (2019). A comparative study of Mohand and Sumudu transforms. Journal of Emerging Technologies and Innovative Research, 6(3), 145-153.

Aggarwal, S., Mishra, R., & Chaudhary, A. (2019). A comparative study of Mohand and Elzaki transforms. Global Journal of Engineering Science and Researches, 6(2), 203-213.

Aggarwal, S., Sharma, S. D., & Vyas, A. (2020). Application of Sumudu Transform for handling Volterra integro-differential equation of first kind with convolution kernel. International Journal of Research and Innovation in Applied Science (IJRIAS), 5(8), 2454-6194.

Akinola, E. I., Olopade, I. A., Akinpelu, F. O., Areo, A. O., & Oyewumi, A. A. (2016). Sumudu Transform Series Decomposition Method for solving nonlinear Volterra integro-differential equations. International Journal of Innovation and Applied Studies, 18(1), 262-267.

Aregbesola, Y. (2003). Numerical solution of Bratu problem using the method of weighted residual. Electronic Journal of the South African Mathematical Society, 3(1), 1-7.

Aris, R. (1975). The mathematical theory of diffusion and reaction in permeable catalyst (Vol. 1). Clarendon Press, Oxford University Press.

Ascher, U. M., Mattheij, R. M. M., & Russell, R. D. (1994). Numerical solution of boundary value problems for ordinary differential equations (1st ed.). SIAM.

Bebernes, J., & Eberly, D. (1989). Mathematical problems from combustion theory. Springer-Verlag.

Chandrasekhar, S. (1967). Introduction to the study of stellar structure. Dover.

Changqing, Y., & Jianhua, H. (2013). Chebyshev wavelets method for solving Bratu's problem. Boundary Value Problems, 2013(1), 1-9.

Cohen, N., & Benavides, J. (2010). Exact solutions of Bratu and Liouville equations. CNMAC, 750-756.

Ezekiel, A. (2013). New improved variational homotopy perturbation method for Bratu-type problems. American Journal of Computational Mathematics, 3(2), 110-113.

Fenta, H. B., & Derese, G. A. (2019). Numerical solution of second order initial value problems of Bratu-type equations using sixth order Runge-Kutta seven stages method. International Journal of Computer Science and Applied Mathematics, 5(1).

Hassan, H. N., & Semary, M. S. (2013). Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method. Communications in Numerical Analysis, 1-14.

Naunyal, G. R., Kumar, U., & Verma, D. (2022). A study on properties and applications of Elzaki transformation. IRE Journals, 5(11), 2456-8880.

Salem, S. A., & Thanoon, T. Y. (2022). On solving Bratu's type equation by perturbation method. International Journal of Nonlinear Analysis and Applications, 13, 2008-6822. http://dx.doi.org/10.22075/ijnaa.2022.6000

Saravi, M., Hermann, M., & Kaiser, D. (2013). Solution of Bratu's equation by He's variational iteration method. American Journal of Computational and Applied Mathematics, 3(1), 46-48.

Wazwaz, A. M. (2000). Solitary wave solutions for the modified KdV equation by Adomian decomposition. International Journal of Applied Mathematics, 3, 361-368.

Wazwaz, A. M. (2005). Adomian's decomposition method for a reliable treatment of the Bratu-type equations. Applied Mathematics and Computation, 652-663.

Wazwaz, A. M. (2016). The successive differentiation method for solving Bratu equation and Bratu-type equations. Romanian Journal of Physics, 61.

Zarebnia, M., & Hoshyar, M. (2014). Solution of Bratu-type equation via spline method. Acta Universitatis Apulensis, 37, 61-72.