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Lorentz Group Action on Ellips Space

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Abstract

The ellips space $\mathcal{E}$ has been constructed as cartesian product $\mathbb{R}^+ \times \mathbb{R}^+ \times [\frac{\pi}{2}, \frac{\pi}{2}]$. Its elements, (a, b, θ), is called as an ellipse with eccentricity is $\epsilon = \sqrt{1 - b^2/a^2}$ if $b < a$ and is $\epsilon = \sqrt{1 - a^2/b^2}$ if $a > b$. The points $(a, b, \pi/2)$ is equal to $(b, a, 0)$. The action of subgrup $SO_{oz}(3,1)$ of Lorentz group $SO_o(3,1)$, containing Lorentz transformations on $x-y$ plane and rotations about z axes, on $\mathcal{E}$ is defined as Lorentz transformation or rotation transformation of points in an ellipse. The action is effective since there are no rigid points in $\mathcal{E}$. The action is also not free and transitive. These properties means that Lorentz transformations change any ellips into another ellips. Although mathematically we can move from an ellipse to another one with the bigger eccentrity but it is imposible physically. This is occured because we do not include the speed parameter into the definition of an ellipse in $\mathcal{E}$.

***Keywords* :** ellips; Lorentz transformation, rotation

1 Introduction

Wirdah, et all has showed that there is one-to-one correspondence between the speed v of a frame of reference and the eccentricity ϵ of an ellipse. Their work was based on the Lorentz contraction of special relativity theory.

Let $K(x, y, z)$ be an inertial frame of reference and $K'(x', y', z')$ be the another one that move with speed v relative to K where x' axes is always coincide with x axes. We denote $\Lambda(v, 0)$ as a Lorentz transformation between K and K'. If K' is move relatif to K such that the y' axes is always coincide with y axes, then the Lorentz transformation between both frame will be denoted by $\Lambda(v, \pi/2)$. More generally, if the x' axes is always coincide with a line forming angle θ with respect to x axes, then the Lorentz transformation will be denoted by $\Lambda(v, \theta)$ which can be obtained by composing $\Lambda(v, 0)$ and $R(\theta)$, that is

$$\Lambda(v, \theta) = \Lambda(v, 0)R(\theta), \tag{1}$$

where $R(\theta)$ is a rotation anticlockwise by an angle θ on $x - y$ plane.

A circle of radius r laying on $x - y$ plane according to K can be transformed into an ellipse with major axes is r and minor axes is r/γ, where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2}$$

by using Lorentz transformation $\Lambda(v, \theta)$, where $v > 0$ and $0 \leq \theta \leq 2\pi$. Note that the vector v must lay on the same plane with the circle's one. The ellipse is actually what will be observed by observers in K'.

Conversely, an ellipse laying on $x - y$ plane whose major axes a parallel to x axes and minor axes b parallel to y axis can be transformed into a circle of radius b

by using Lorentz transformation $\Lambda(v,0)$. If the major axes a is parallel to y axes and the minor axes b parallel to x axes then Lorentz transformation $\Lambda(v,\pi/2)$ will transforms the ellipse into a circle of radius b.

More generally, if K' is moving along a line forming angle θ with respect to x axes, then the ellipse laying on K, whose major axes a is parallel to x axes, will be observed as a circle of radius

$$r = \frac{1}{\sqrt{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}}}, \tag{3}$$

provided $\theta = \pm\pi/4$. The ellips will still be observerd as ellips in K' if θ is satisfying $-\pi/4 < \theta < \pi/4$ and speed v is satisfying $v > v_c$, where

$$\begin{aligned} v_c &= c\sqrt{1-\frac{A}{B}}, \\ A &= \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}, \\ B &= \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}. \end{aligned} \tag{4}$$

If the major axes is parallel to y', then the ellipse laying on K will be observed as a circle of radius r as given in eq.(3) provided θ is satisfying $\theta < -\pi/4$ or $\theta > \pi/4$ and speed v is satisfying $v < v_c$.

2 Ellips Space

According to those results, we can construct an ellips space which contains all ellips in a plane including circles. Let we call $\mathcal{E} \equiv \mathbb{R}^+ \times \mathbb{R}^+ \times [\frac{-\pi}{2}, \frac{\pi}{2}] \subset \mathbb{R}^3$, where $\mathbb{R}^+$ contains only real numbers having positive values. The θ in any $(a,b,\theta) \in \mathcal{E}$ is defined as an angle between line of length a and x axis.

Every $(a,b,\theta) \in \mathcal{E}$ will be called as an ellipse with eccentricity

$$\epsilon = \sqrt{1-\frac{b^2}{a^2}} \tag{5}$$

provided $b < a$. If $b > a$, then the eccentricity is given by

$$\epsilon = \sqrt{1-\frac{a^2}{b^2}}. \tag{6}$$

The special cases is when $a = b$, so that (a,a,θ) will be called as a circle. It is clear that $(a,b,\pi/2) = (b,a,0)$.

3 Action of Lorentz Group on The Ellips Space

Since ellips space contains only ellips, including circles, laying in $x-y$ plane, we only need to deal with subgroup of Lorentz group $SO_o(3,1)$ which contains Lorentz transformations on $x-y$ plane and rotation about z axes.

Let we denote $SO_{oz}(3,1)$ be a set of all Lorentz transformation on $x-y$ plane and all rotations about z axes. It is clear that $SO_{oz}(3,1)$ is a subgroup of $SO_o(3,1)$ since composition of any two Lorentz transformations on $x-y$ plane is the same as the composotion of another Lorentz transformation in $x-y$ plane and a rotation about z axes, that is[3]

$$\Lambda(v_1,0)\Lambda(v_2,\pi/2) = R_z(\theta)\Lambda(v_3,\theta). \tag{7}$$

Now we define an action of $SO_{oz}(3,1)$ on $\mathcal{E}$ as follows

$$\begin{aligned} \alpha :& SO_{oz}(3,1) \times \mathcal{E} \to \mathcal{E} \\ & \Lambda(v,\phi) \times (a,b,\theta) \mapsto \alpha(\Lambda(v,\phi),(a,b,\theta)) \end{aligned} \tag{8}$$

where $\alpha(\Lambda(v,\phi),(a,b,\theta))$ is defined as a transformation of any ellips into another ellips by Lorentz transformations in $SO_{oz}(3,1)$.

Kernel of action α is

$$ker(\alpha) = \{\mathbb{I}_4\}, \tag{9}$$

which mean that any ellips should change into another ellips due to Lorentz transformations. Moreover, since $(0,0,0)$ is not contained in $\mathcal{E}$ then it is clear that there are no rigid points in $\mathcal{E}$. Therefore we can say that the action of α is effective.

The set of all rotations about z axes is a stabilizer subgroup of $SO_{oz}(3,1)$ with respect to points having form (a,a,θ). This means that the action α is not free.

Let $\Lambda(v,\theta)$ transform (a_1,b_1,θ_1) into (a_2,b_2,θ_2). Then the inverse of $\Lambda(v,\theta)$, i.e. $\Lambda^{-1}(v,\theta)$, will transform (a_2,b_2,θ_2) into (a_1,b_1,θ_1). From this fact it is clear that the action of α is transitive.

4 Conclusion

Since the action of $SO_{oz}(3,1)$ on $\mathcal{E} = \mathbb{R}^+ \times \mathbb{R}^+ \times [\frac{\pi}{2}, \frac{\pi}{2}]$ is transitive, then any ellipse can be viewed mathematically as any other ellips with different eccentricity, even with the bigger one. However, physically, it is imposible to view an ellipse as another ellipse with the bigger eccentricity. This is occured because we do not include the speed parameter into the definition of an ellipse in $\mathcal{E}$. The transitivity of the action is also means that Lorentz transformation change the geometry of objects, especially ellips.

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