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Modified Variational Iteration Method with Chebyshev Polynomials for Solving 12th order Boundary Value problems

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Abstracts

We consider in this paper an illustration of the modified variational iteration method (MVIM) as an effective and accurate solver of 12th order boundary value problem (BVP). For this reason, the Chebyshev polynomials of the principal kind was utilized as a premise capabilities in the guess of the logical capability of the given issue. The strategy is applied in an immediate manner without utilizing linearization or irritation. The subsequent mathematical confirmations recommend that the strategy is without a doubt successful and exact as applied to a few direct and nonlinear issues as mathematical trial and error. Maple 18 was used for all computational simulations carried out in this research.

Keywords: Boundary value problems; Chebyshev polynomials of the first kind; variational iteration method; basis functions

1. Introduction

Let consider the n-th boundary value problem of the form

$$\sum_{r=0}^{n} a_{n-r} \frac{d^{n-r}y}{dx^{n-r}} = f(x), \qquad (1)$$

subject to boundary conditions

$$y^{(n)}(a) = A_i \, , i = 0(1)n \, ,$$

$$y^{(n)}(b) = B_i \, , i = 0(1)n \, ,$$

where $a_{n-r}, r = 0(1)n$ are unknown parameters to be determined, , A_i *and* B_i, $i = 0(1)n$ are real constant, , $f(x)$ continuous on [a,b], and n denotes the nth order derivative. Equations of the form (1) are relevant in many area of real-life applications such as heat transfer, viscoelastic flow, Abrasive blasting, control theory, etc. Thus, solving these problems is of huge interest to researchers. Over the years, there had been several iterative approaches in solving these problems. This is because the iterative approaches present the solution in a rapidly convergent seires. Modification of the variational iteration technique can be carried out to huge class of the linear and non-linear differential equations [1].

For lawsuit, Ojobor et al. [2] habitual a inclined variational simulate propose to helter-skelter accounting Polynomials to direct the numerical undertake responsibility for to eight conduct oneself block report problems. Way development the numerical support of fifth counterfeit ditch profit company permit Mamadu-Njoseh polynomials as grief functions [3]. Token to the talents train guestimate make advances for a outspoken BVP [4]. Including, the path of tau and tau-collocation determine overtures was very much old by Njoseh and Mamadu [5] to have designs on the fill of prime and shoved ordinary differential equations. Previolus hamper [6-9] adopted the variational parrot fractionation path and the variational replication homotopy butterflies in the stomach proposals for the numerical defence of high-class work boundary-line value problems.

Mohyud-Din et al. [10] applied the variational iterative scheme to achieve the numerical solution of 12th order boundary value problem. Mirmoradi et al. [11] applied the homotopy irritation technique to look for the mathematical arrangement of twelfth request limit esteem issue. In draught exercise by Mohyud-Din and Yildirim [12] false the homotopy terrifying draw close by the flag variational parrot approach to pointing the numerical conform to of ninth and tenth-order impediment report sway. Arshed and Hussain [13] hand-me-down the homotopy apprehend near and the variational emulate sound out to come into possession of the numerical declaration of seventh-order block standing oppression.

Shahid and Iftikhar [14] used the Collocation technique to achieve the numerical answer of sixth-order boundary value problems. Caglar et al. [15] likewise looks for the mathematical arrangement of fifth request limit esteem issue with 6th degree B-spline. The Adominan decomposition method was used by Adominan [16] to solve both linear and nonlinear BVPs. Also, Hesameddini and Rahimi [17] apllied modification of the reconstruction of variational new release technique for fixing multi 12–order fractional differential equation. Moreover, Chebyshev Polynomial can used for basic to Galerkin method for nonlinear higher-order boundary value problems [18].

In this paper, we proposed a changed rendition of the variational emphasis strategy for the arrangement of 12th order boundary value problems. In this method, the correction functional is first built for the expected BVP, and the Lagrange multiplier is figured ideally by means of the variational hypothesis. The Chebyshev polynomials are then utilized as preliminary capabilities. The strategy was applied to both straight and nonlinear BVPs and the subsequent mathematical proof show that it is compelling, precise and solid as contrasted and existing technique as accessible in literature.

2. The Standard Variational Iteration Method

Let considered the differential equation of the form

$$Lu + Nu - g(x) = 0, \tag{2}$$

subject to some auxillary conditions, where L is a linear operator, N a nonlinear operator and $g(x)$ is the source term [19].

Developing the amendment practical by variational cycle technique, the issue (2) is given bythe correction functional by

$$u_{n+1} = u_n(x) + \int_0^x \lambda(s)\left[Lu_n(s) + N\widetilde{u_n(s)} - g(s)\right]ds, \tag{3}$$

where $\lambda(s,)$ is a Lagrange multiplier obtained optimally using variational theory, $\widetilde{u_n(t)}$is considered as a restricted variations. i.e. $\widetilde{u_n}= 0.$

The Lagrange Multiplier plays a clever establishment in the pillar of the retort of the dealing. Reckoning, we synopsize the essay for the Lagrange Multiplier singly

$$\lambda(s) = (-1)^m \frac{1}{(m-1)!}(s-x)^{n-1}, \tag{4}$$

where m is the highest order of the given BVP.

3. Result and Discussion

The Chebyshev polynomials of the first kind was proposed by Lvovich Chebyshev with the weight function

$$w(x) = (1-x^2)^{-1/2}, x \in [-1,1].$$

The construction of the Chebyshev polynomials of the first kind was based on the following three properties:

(i) $T_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r$

(ii) $\langle T_m(x), T_n(x)\rangle = 0, m \neq n$

(iii) $T_n(x) = 1$

where $T_n(x), i = 0,1,2,3 \ldots.$ are the Chebyshev polynomials of the first kind [20].

Hence, the first six Chebyshev polynomials of the first kind are generated with the aid of Maple 18, and are given below:

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 - 1$

$T_5(x) = 16x^5 - 20x^3 - 3x$

The shifted chebyshev polynomial can be obtained from a simple recursive relation

$$T_{n+1}^*(x) = 2\left[\frac{2(x-a)}{b-a} - 1\right] T_n^*(x) - T_{n-1}^*(x) \tag{5}$$

where $T_n^*(x), n = 0,1,2,3, \ldots$ are called the shifted Chebyshev polynomials defined in the interval [a,b]. Thus, the first few shifted Chebyshev polynomials are as follows:

$T_0^*(x) = 1$

$T_1^*(x) = 2x - 1$

$T_2^*(x) = 8x^2 - 8x + 1$

$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$

$T_4^*(x) = 128x^4 - 48x^3 + 160x^2 - 32x + 1$

$$T_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

$$T_6^*(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 70x + 1$$

Using (2) and (3), we now assume an approximate solution of the form

$$u_{n,N}(x) = \sum_{i=0}^{N} a_{i,N} T_{i,N}^*(x)$$

Where $T_{i,N}(x)$ are Shifted chebyshev Polynomials, are constants to be determined, and the degree of approximant. Hence, we obtain the following iterative scheme

$$u_{n+1,N} = \sum_{i=0}^{N} a_{i,N} T_{i,N}^*(x) + \int_0^x \lambda(t)\left[L \sum_{i=0}^{N} a_{i,N} T_{i,N}^*(x) + N \sum_{i=0}^{N} a_{i,N} T_{i,N}^*(x) - g(t)\right] dt \tag{5}$$

In this room we functional the in name only make a proposal to to interpret examples of Twelfth role of BVPs. The candid focus to is to make out these duo examples buy the MVIM tending in block 5 and consider our benefits relating to the presented calculation in [10] and [11].

Example 1 [10]

Consider the following 12th order linear boundary value problem

$$u^{(xii)}(x) = -xu(x) - e^x(120 + 23x + x^3), 0 \le x \le 1 \quad (6)$$

with boundary conditions

$$u(0) = 0, u'(0) = 1, u''(0) = 0, u'''(0) = -3, u^{iv}(0) = -8,$$

$$u^v(0) = -15 u(1) = 0, u'(1) = -e, u''(1) = -4e, \quad (7)$$

$$u'''(1) = -9e, u^{iv}(1) = -16, u^v(1) = -25e$$

The solution is

$$u(x) = x(1-x)e^x \quad (8)$$

The correct functional for the boundary value problem

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left[\frac{d^{12}}{dt^{12}}u_n(t) - tu_n(t) + e^t(120 + 23t + t^3)\right]dt$$

Making the correct functional stationary, $\lambda(t) = \frac{(t-x)^{11}}{11!}$ as the Lagrange multiplier, we have the following

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(t-x)^{11}}{11!}\left[\frac{d^{12}}{dt^{12}}u_n(t) - tu_n(t) + e^t(120 + 23t + t^3)\right]dt$$

Applying the proposed method, we assume an approximate solution of the form

$$u_{n,12}(x) = \sum_{i=0}^{12} a_{i,12}T^*_{i,12}(x)$$

$$u_{n+1,N}(x) = \sum_{i=0}^{12} a_{i,12}T^*_{i,12}(x) + \int_0^x \frac{(t-x)^{11}}{11!}\left[\frac{d^{12}}{dt^{12}}\left(\sum_{i=0}^{12} a_{i,12}T^*_{i,12}(t)\right) - t\left(\sum_{i=0}^{12} a_{i,12}T^*_{i,12}(t)\right) + e^t(120 + 23t + t^3)\right]dt$$

Iterating and applying the boundary circumstance in Equation (7) the values of the unknown constants may be determined as follows

$a_{0,12} = 0.2125978650$, $a_{1,12} = 0.05260425742$, $a_{2,12} = -0,2060564051$, $a_{3,12} = -0.05206095678$

$a_{4,12} = -0.006507710570$, $a_{5,12} = -0.00054175702$, $a_{6,12} = -0.000033828688$

$a_{7,12} = -0.000001691207$, $a_{8,12} = -0.0000000707301$, $a_{9,12} = -0.0000000025836$, $a_{10,12} = -0.000000000088608$

$a_{11,12} = -0.00000000000304$, $a_{12,12} = -0.00000000000006467$

Finally, the series solution is

$$u(x) = x - 0.4999999956x^3 - 0.33333304x^4 - 0.12500003x^5 - 0.03333312x^6 - 0.00694531x^7 - 0.00118815x^8 - 0.0001765546x^9 - 0.0000200672x^{10} - 0.0000030702x^{11} + O(x)^{12}$$

Table 1. The result of the proposed method compared with Homotopy

x	Exact solution	Approximate Solution	Present Method Error	HPM Error
0.0	0.0000000	0.0000000	0.0000E+00	0.0000E+00
0.1	0.0994654	0.0994654	2.0000E-11	3.0000E-11
0.2	0.1954244	0.1954244	1.0000E-10	0.0000E+00
0.3	0.2834708	0.2834708	2.0000E-10	1.0000E-10
0.4	0.3580379	0.3580379	2.0000E-10	2.0000E-10
0.5	0.4121803	0.4121803	3.0000E-10	1.1000E-09
0.6	0.4373085	0.4373085	0.0000E+00	4.4000E-09
0.7	0.4228881	0.4228881	4.0000E-10	1.3500E-08
0.8	0.3560865	0.3560866	2.0000E-10	3.6800E-08
0.9	0.2213643	0.2213644	7.0000E-10	9.0100E-08
1.0	0.0000000	0.0000000	6.2108E-10	2.0270E-07

The result of the proposed method compared with Homotopy Perturbation Method is shown in Table 1.

Example 2 [10]

Consider the twelfth order boundary value problem of the form

$$u^{(xii)}(x) = 2e^{-x}u^2(x) + u'''(x), 0 \leq x \leq 1, \quad (9)$$

With boundary conditions

$$u(0) = u''(0) = u^{(iv)}(0) = u^{(vi)}(0) = u^{(viii)}(0) = u^{(x)}(0) = 1,$$

$$u(1) = u''(1) = u^{(iv)}(1) = u^{(vi)}(1) = u^{(viii)}(1) = u^{(x)}(1) = e^{-1} \quad (10)$$

The exact solution is

$$u(x) = e^{-x} \quad (11)$$

The correction functional for the boundary value problem (9) and (10) is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left[\frac{d^{12}}{dt^{12}}u_n(t) - 2e^{-t}u_n^2(t) - u'''(t)\right]d$$

Making the correction functional stationary using, $\lambda(t) = \frac{(t-x)^{11}}{11!}$ as the Lagrange multiplier, hence we get the following iterative scheme

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(t-x)^{11}}{11!}\left[\frac{d^{12}}{dt^{12}}u_n(t) - 2e^{-t}u_n^2(t) - u'''(t)\right]dt$$

Applying the modified variational iteration method using Chebyshev polynomials, we assume an approximate solution of the form

$$u_{n,12}(x) = \sum_{i=0}^{12} a_{i,12}T^*_{i,12}(x)$$

Hence, we have the following iterative formula

$$u_{n+1,N}(x) = \sum_{i=0}^{12} a_{i,12}T^*_{i,12}(x) + \int_0^x \frac{(t-x)^{11}}{11!}\left[\frac{d^{12}}{dt^{12}}\left(\sum_{i=0}^{12} a_{i,12}T^*_{i,12}(t)\right) - 2e^{-t}\left(\sum_{i=0}^{12} a_{i,12}T^*_{i,12}(t)\right)^2 - \frac{d^3}{dt^3}\left(\sum_{i=0}^{12} a_{i,12}T^*_{i,12}(t)\right)\right]dt$$

Table 2. The error of the proposed method compared with Variational Iteration Method.

X	Exact solution	Approximate Solution	Present Method Error	VIM Error
0.0	1.0000000	1.0000000	0.0000	0.0000
0.1	0.9048374	0.9048372	1.51E-7	1.61E-7
0.2	0.8187308	0.8187303	2.03 E-7	3.07 E-7
0.3	0.7408182	0.7408175	3.92 E-7	4.22 E-7
0.4	0.6703200	0.6703192	4.14 E-7	4.97 E-7
0.5	0.6065307	0.6065298	4.56 E-7	5.22 E-7
0.6	0.5488116	0.5488108	3.92 E-7	4.97 E-7
0.7	0.4965853	0.4965846	3.92 E-7	4.22 E-7
0.8	0.4493290	0.4493285	3.02 E-7	3.07 E-7
0.9	0.4065697	0.4065694	1.54 E-7	1.61 E-7
1.0	0.3678794	0.3678794	0.0000	2.00 E-10

Emphasizing and applying the limit condition in condition (10) the upsides of the obscure constants not set in stone as follows

$$a_{0,12} = 0.6450348671,\quad a_{1,12} = -0.3128416055,\quad a_{2,12} = 0{,}03870454340,\quad a_{3,12} = -0.003208682549$$

$$a_{4,12} = 0.0001998955731,\quad a_{5,12} = -0.000009975072982,\quad a_{6,12} = 0.0000004155788$$

$$a_{7,12} = -0.0000000147895,\quad a_{8,12} = 0.000000000460623,\quad a_{9,12} = -0.000000000012044,\quad a_{10,12} = 0.000000000000465281$$

$$a_{11,12} = 0.0000000000000016481,\quad a_{12,12} = 0.000000000000000038330$$

Thus, the series arrangement is given as and the relating results are displayed in table 2.

$$1.000000000 - 1.000002688x + 0.500000001x^2 - 0.1666622435x^3 + 0.0416666667x^4 - 0.0083355196x^5 + 0.00138888887x^6 - 0.0001978951x^7 + 0.000024801587x^8 - 2.829495 \times 10^{-6}\, x^9 - 2.755732 \times 10^{-7} x^{10} + O(x)^{11}$$

4. Conclusion

In this paper, the adjusted variational cycle strategy utilizing Chebyshev polynomials has been utilized effectively to get the mathematical arrangements of twelfth request limit esteem issues. The alteration includes the utilization of chebyshev polynomials as preliminary capability combined with standard variational iterative method. The technique is applied in an immediate manner without utilizing linearization or irritation. It could be reasoned that the proposed MVIM is an exceptionally strong and productive technique in finding the scientific answers for the class of issue considered.

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