JOURNAL OF
NATURAL SCIENCES
AND
MATHEMATICS RESEARCH

Available online at http://journal.walisongo.ac.id/index.php/jnsmr

Quaternionic Version of Rotation Groups

Latief Rahmawati
Ma'had Ta'dzhiimussunnah, Semarang, Indonesia

Muhammad Ardhi Khalif
Jurusan Pendidikan Fisika, FITK UIN Walisongo, Semarang, Indonesia

Muhammad Farchani Rosyid
Jurusan Fisika, FMIPA Universitas Gadjah Mada, Yogyakarta, Indonesia

Corresponding author :
uda_ardhi@yahoo.com
Telp. (+62-24-7601295)
Avaliable online :
27 June 2015

Abstract

Quaternionic version of rotation group $SO(3)$ has been constructed. We construct a quatenionic version of rotation operation that act to a quaternionic version of a space coordinate vector. The computation are done for every rotation about each coordinate axes (x,y, and z). The rotated quaternionic space coordinate vector contain some unknown constants which determine the quaternionic rotation operator. By solving for that constants, we get the expression of the quaternionics version of the rotation operator. Finally the generators of the group were obtained. The commutation of the generator were also computed.

***Keywords* :** quaternion; rotation group

1 INTRODUCTION

Quaternion algebra $\mathbb{H}$ was invented by W. R. Hamilton in October 16th 1843. It is a 4-dimensional number system expressed in $1, i, j$ and k satisfying

$$ii = jj = kk = -1. \tag{1}$$

A real quaternionic can be written in the expression of two complex numbers through simplectic decomposition

$$q = a_0 + ia_1 + ja_2 + ka_3 = \xi + j\zeta, \qquad a_{0,1,2,3} \in \mathbb{R}, \tag{2}$$

where $\xi = a_0 + ia_1 \in \mathbb{C}$ and $\xi = a_2 - ia_3 \in \mathbb{C}$. The expression $q = \xi + j\zeta$ is called as **quaternionic spinor**.

Total conjugate of q is given by

$$q^\dagger = a_0 - ia_1 - ja_2 - ka_3, \tag{3}$$

or may be written as

$$q^\dagger = \xi^* - j\zeta. \tag{4}$$

For every quaternion q, we can make an anti-hermitian quaternion $qiq^\dagger$ which may be related to a point (x, y, z) in 3 dimensional space through

$$\begin{aligned} qiq^\dagger &\equiv ix + jy + kz, \\ x &\equiv |\xi|^2 - |\zeta|^2, \\ y - iz &\equiv 2i\zeta\xi^*. \end{aligned} \tag{5}$$

Hence we get

$$\begin{aligned} x &= (a_0^2 + a_1^2) - (a_2^2 - a_3^3), \\ y &= 2(a_0 a_3 + a_1 a_2), \\ z &= 2(a_1 a_3 - a_0 a_2). \end{aligned} \tag{6}$$

Although it was abandoned by some people, but lately the research in quaternion theory was developed again. Among who devotes his/her research in quaternion theory is Stefano De Leo. He has introduced some quaternionic group with its generator (de Leo, 1999). He showed some group generators without showing its derivation.

In this paper we will derive generators and its commutation relations of quaternionic version of rotation groups and Lorentz group directly from the definition of the groups respectively. The groups that will be discussed in this paper is quaternionic version of rotation groups $U(2)$, $SU(2)$, $O(3)$, $SO(3)$ and quaternionic version of Lorentz groups $SO_o(3,1)$, $SL(2,\mathbb{C})$.

2 $\mathbb{R}.\mathbb{C},\mathbb{H}-$RIGHT LINEAR OPERATORS

Before we discuss about quaternionic groups, we need to introduce about quaternionic operators and the space containing it. There are two kind of operator action, left and right action. Due to uncommutativity of quaternionic multiplication, we need to pay attention to the distinction of the two operators.

Left action of $1, i, j, k$ is represented by action of left operator

$$L_\mu \equiv (1, \vec{L}), \qquad \vec{L} = (L_i, L_j, L_k) \tag{7}$$

on $q \in \mathbb{H}$, while its right action is repesented by action of right operator

$$R_\mu \equiv (1, \vec{R}), \qquad \vec{R} = (R_i, R_j, R_k). \tag{8}$$

The action of L_μ and R_μ are defined as

$$L_\mu : \mathbb{H} \to \mathbb{H}, \quad L_\mu q \equiv h_\mu q \in \mathbb{H} \tag{9}$$

$$R_\mu : \mathbb{H} \to \mathbb{H}, \quad R_\mu q \equiv q h_\mu \in \mathbb{H} \tag{10}$$

respectively, where $h_\mu \equiv (1, i, j, k)$. Both operators satisfies the following relation

$$L_i^2 = L_j^2 = L_k^2 = L_i L_j L_k = -1 \tag{11}$$

$$R_i^2 = R_j^2 = R_k^2 = R_i R_j R_k = -1 \tag{12}$$

and the following commutation relation

$$[L_i, R_i] = 0, \quad [L_j, R_j] = 0, \quad [L_k, R_k] = 0. \tag{13}$$

We need to introduce another operator that we called as barred operator $|$ and is defined by

$$(A|B)q \equiv AqB, \qquad A, B, q \in \mathbb{H} \tag{14}$$

In addition to the barred operator, the simultaneous left and right action can be represented by

$$\begin{aligned} M_{\mu\nu} &\equiv L_\mu \otimes R_\nu \\ M_{\mu\nu} q &\equiv (L_\mu \otimes R_\nu) q = h_\mu q h_\nu. \end{aligned} \tag{15}$$

Moreover, we alse define the following sets

$$\begin{aligned} \mathbb{H}^L &\equiv \{a^\mu L_\mu | a^\mu \in \mathbb{R}\} \\ \mathbb{H}^R &\equiv \{b^\mu R_\mu | b^\mu \in \mathbb{R}\} \\ \mathbb{H}^L \otimes \mathbb{H}^R &\equiv \{a^{\mu\nu} L_\mu \otimes R_\nu | a^{\mu\nu} \in \mathbb{R}\} \end{aligned} \tag{16}$$

and it is clear that

$$\begin{aligned} \mathbb{H}^L &\cong \mathbb{H}^L \otimes \{1\}, \\ \mathbb{H}^R &\cong \{1\} \otimes \mathbb{H}^R. \end{aligned} \tag{17}$$

The addition operation in $\mathbb{H}^L \otimes \mathbb{H}^R$ is given by

$$\begin{aligned} (a^{\mu\nu} L_\mu \otimes R_\nu + b^{\sigma\rho} L_\sigma \otimes R_\rho) q =& (a^{\mu\nu} L_\mu \otimes R_\nu) q \\ &+ (b^{\sigma\rho} L_\sigma \otimes R_\rho) q \end{aligned} \tag{18}$$

and the multiplication operation is given by

$$a(a^{\mu\nu} L_\mu \otimes R_\nu) q = (a a^{\mu\nu} L_\mu \otimes R_\nu) q. \tag{19}$$

For the sake of simplicity, we introduce the notation

$$\mathcal{O}_\mathbb{X} : \mathbb{H} \to \mathbb{H} \tag{20}$$

to represents elements in $\mathbb{H}^L \otimes \mathbb{H}^R$ that are right linear with respect to field $\mathbb{X}$. It follows that operators $\mathcal{O}_\mathbb{R}$ in $\mathbb{H}^L \otimes \mathbb{H}^R$ of the form

$$\mathcal{O}_\mathbb{R} = a^{\mu\nu} L_\mu \otimes R_\nu \in \mathbb{H}^L \otimes \mathbb{H}^R \tag{21}$$

are right $\mathbb{R}$-linear operators and satisfy

$$M_{\mu\nu}(q\rho) = (M_{\mu\nu} q)\rho = \rho(M_{\mu\nu} q), \qquad \rho \in \mathbb{R}. \tag{22}$$

The set of all $\mathcal{O}_\mathbb{R}$ is clearly equal to $\mathbb{H}^L \otimes \mathbb{H}^R$. If $\mathbb{X} = \mathbb{C}$, then $\mathcal{O}_\mathbb{C}$ in $\mathbb{H}^L \otimes \mathbb{H}^R$ is of the form

$$\mathcal{O}_\mathbb{C} = a^{\mu n} L_\mu \otimes R_n, \quad n = 0, 1 \tag{23}$$

because it satisfy

$$M_{\mu n}(q\xi) = (M_{\mu n} q)\xi, \qquad \xi \in \mathbb{C}. \tag{24}$$

The set of all $\mathcal{O}$ as in (23) will be denoted by $\mathbb{H}^L \otimes \mathbb{C}^R$.

Due to the associativity of quaternion multiplication, operator $\mathcal{O}_\mathbb{H}$ defined by

$$\mathcal{O}_\mathbb{H} = a^\mu L_\mu \in \mathbb{H}^L \tag{25}$$

is a right linear operator with respect to $\mathbb{H}$ in $\mathbb{H}$ since it follows that

$$L_\mu(q_1 q_2) = (L_\mu q_1) q_2, \qquad q_{1,2} \in \mathbb{H}. \tag{26}$$

In the last definition, we have used abuse of notation because $\mathbb{H}$ is not a field. Hence it follows that

$$\mathbb{H}^L \subset \mathbb{H}^L \otimes \mathbb{C}^R \subset \mathbb{H}^L \otimes \mathbb{H}^R. \tag{27}$$

The direct product of two right $\mathbb{R}$-linear

$$\mathcal{O}^a_{\mathbb{R}} = a^{\mu\nu} M_{\mu\nu}, \qquad \text{and} \qquad \mathcal{O}^b_{\mathbb{R}} = b^{\tau\sigma} M_{\tau\sigma}, \tag{28}$$

is defined by

$$\mathcal{O}^a_{\mathbb{R}}\mathcal{O}^b_{\mathbb{R}} = a^{\mu\nu} b^{\tau\sigma} L_\mu L_\tau \otimes R_\sigma R_\nu. \tag{29}$$

The conjugation operation for left and right operators are defined by

$$\begin{aligned} L^\dagger_\mu &= L^\mu \equiv -\eta^{\mu\nu} L_\nu, \\ R^\dagger_\mu &= R^\mu \equiv -\eta^{\mu\nu} R_\nu, \end{aligned} \tag{30}$$

while conjugation operation for $\mathcal{O}_{\mathbb{R}}$ is defined by

$$\mathcal{O}^\dagger_{\mathbb{R}} \equiv a^{\mu\nu} L^\dagger_\mu \otimes R^\dagger_\nu = a^{\mu\nu} L^\mu \otimes R^\nu \in \mathbb{H}^L \otimes \mathbb{H}^R. \tag{31}$$

Hence, the conjugate of eq.(29) is

$$\begin{aligned} (\mathcal{O}^a_{\mathbb{R}}\mathcal{O}^b_{\mathbb{R}})^\dagger &= a^{\mu\nu} b^{\tau\sigma} (L_\mu L_\tau \otimes R_\sigma R_\nu)^\dagger \\ &= a^{\mu\nu} b^{\tau\sigma} (L_\mu L_\tau)^\dagger \otimes (R_\sigma R_\nu)^\dagger \\ &= a^{\mu\nu} b^{\tau\sigma} L^\dagger_\tau L^\dagger_\mu \otimes R^\dagger_\nu R^\dagger_\sigma \\ &= {\mathcal{O}^b_{\mathbb{R}}}^\dagger {\mathcal{O}^a_{\mathbb{R}}}^\dagger. \end{aligned} \tag{32}$$

3 GROUP $U(1, \mathbb{H}^L)$

In this section, we will construct the quaternionic version of rotation group $SO(3)$. We start by defining a set $U(1, \mathbb{H}^L)$ containing elements of $\mathbb{H}^L$ satisfying

$$uu^\dagger = u^\dagger u = 1, \qquad u \in \mathbb{H}. \tag{33}$$

As indicated in the definition, the set naturally get a multiplication operation inherited from those in $\mathbb{H}^L$. It is easy to check that $U(1, \mathbb{H}^L)$ is a group with the given multiplication operation. For every $v \in U(1, \mathbb{H}^L)$, its inverse $v^{-1} = v^\dagger$ is satisfy condition (33) also. According to the definition, it is clear that $U(1, \mathbb{H}^L) \subset \mathbb{H}^L$.

Now we define actions of u to quaternion q and its anti-hermitian $qiq^\dagger$ by

$$\begin{aligned} q &\mapsto uq \\ qiq^\dagger &\mapsto uqiq^\dagger u^\dagger. \end{aligned} \tag{34}$$

We want to show that a transformation of $V = (x, y, z)^T \in \mathbb{R}^3$ by an element of $SO(3)$ is equivalen to a transformation of $q = \xi + j\zeta$ by an element of $U(1, \mathbb{H}^L)$. Now we assume that $\tilde{q} = uq$, or more explicitly we can write

$$\tilde{q} = \tilde{\xi} + j\tilde{\zeta} \equiv [A + jB](\xi + j\zeta). \tag{35}$$

Here, A and B are elements of $\mathbb{C}$ such that $u = A + jB$ is an element of $U(1, \mathbb{H}^L)$. From eq.(35), we get

$$\begin{aligned} \tilde{\xi} &= A\xi - B^*\zeta, \\ \tilde{\zeta} &= B\xi + A^*\zeta. \end{aligned} \tag{36}$$

By using eq.(5), we obtain

$$\begin{aligned} \tilde{x} =& (|A|^2 - |B|^2)x - i(AB - A^*B^*)y \\ &+ (AB + A^*B^*)z \\ \tilde{y} - i\tilde{z} =& 2iA^*Bx + A^{*2}(y - iz) + B^2(y + iz). \end{aligned} \tag{37}$$

Now we have to find three pairs of values of A and B such that each $A + jB$ are related to transformations $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ in $SO(3)$, where $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are defined by

$$\mathcal{R}_1(\theta) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{38}$$

$$\mathcal{R}_2(\theta) \equiv \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & & \cos(\theta) \end{pmatrix}, \tag{39}$$

$$\mathcal{R}_3(\theta) \equiv \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

In order to obtain transformation u_x which is related to $\mathcal{R}_1$, the first condition that must be satisfied by A and B is

$$\begin{aligned} |A|^2 - |B|^2 &= 1, \\ AB - A^*B^* &= 0, \\ AB + A^*B^* &= 0, \end{aligned} \tag{41}$$

which gives

$$|A| = 1 \quad \text{dan} \quad B = 0. \tag{42}$$

If we set

$$\begin{aligned} A &= \cos\frac{\theta_x}{2} + i\sin\frac{\theta_x}{2}, \\ B &= 0, \end{aligned} \tag{43}$$

then we obtain, for every $\theta \in \mathbb{R}$, that

$$u_x(\theta) = \cos\frac{\theta_x}{2} + i\sin\frac{\theta_x}{2} = e^{i\theta_x}, \tag{44}$$

is an element of $U(1, \mathbb{H}^L)$ that transforms q equivalently with transformations of $\mathcal{R}_1(\theta)$ to $V \in \mathbb{R}^3$.

In order to obtain the corresponding u_y that related to $\mathcal{R}_2$, the first condition that must be satisfied by A and B is

$$\begin{aligned} A^{*2} + B^2 &= 1, \\ AB - A^*B^* &= 0, \end{aligned} \tag{45}$$

which force both A and B to be real. Furthermore, if we set

$$A = \cos\frac{\theta_y}{2} \quad \text{dan} \quad B = \sin\frac{\theta_y}{2}, \tag{46}$$

which satisfies

$$\begin{aligned} A^2 - B^2 &= \cos\theta_y, \\ 2AB &= \sin\theta_y, \end{aligned} \tag{47}$$

then we obtain

$$u_y(\theta) = \cos\frac{\theta_y}{2} + j\sin\frac{\theta_y}{2} = e^{j\theta_y}, \tag{48}$$

which is an element of $U(1,\mathbb{H}^L)$ that corresponds to $\mathcal{R}_2(\theta) \in SO(3)$.

Finally, the conditions that must be satisfied by A and B in order to get the corresponding u_z is

$$\begin{aligned} A^{*2} - B^2 &= 1 \\ AB + A^*B^* &= 0, \end{aligned} \tag{49}$$

which gives

$$\mathrm{Re}(A) \neq 0 \quad \text{and} \quad \mathrm{Re}(B) = 0. \tag{50}$$

Now, by setting

$$A = \cos\frac{\theta_z}{2} \quad \text{and} \quad B = -i\sin\frac{\theta_z}{2}, \tag{51}$$

which satisfies

$$A^2 - |B|^2 = \cos\theta_z \quad \text{and} \quad 2iAB = \sin\theta_z, \tag{52}$$

we obtain transformation

$$u_z(\theta) = \cos\frac{\theta_z}{2} + k\sin\frac{\theta_z}{2} = e^{k\theta_z} \in U(1,\mathbb{H}^L), \tag{53}$$

which corresponds to $\mathcal{R}_3(\theta) \in SO(3)$.

It is easy to see that all of u_x, u_y, u_z will be an identity $1 \in U(1,\mathbb{H}^L)$ if we set $\theta_x = \theta_y = \theta_z = 0$. Therefore, its differentiation at the identity is given by

$$\begin{aligned} \hat{u}_1 &\equiv \left.\frac{du_x}{d\theta_x}\right|_{\theta_x=0} = \frac{i}{2} = \frac{L_i}{2}, \\ \hat{u}_2 &\equiv \left.\frac{du_y}{d\theta_y}\right|_{\theta_y=0} = \frac{j}{2} = \frac{L_j}{2}, \\ \hat{u}_3 &\equiv \left.\frac{du_z}{d\theta_z}\right|_{\theta_z=0} = \frac{k}{2} = \frac{L_k}{2}. \end{aligned} \tag{54}$$

The above three equations gives us generators of rotations about x, y and z-axes respectively, and satisfy the following commutation relations

$$\begin{aligned} \hat{u}_1 &= [\hat{u}_2, \hat{u}_3], \\ \hat{u}_2 &= [\hat{u}_3, \hat{u}_1], \\ \hat{u}_3 &= [\hat{u}_1, \hat{u}_2]. \end{aligned} \tag{55}$$

Now we can express every u element in $U(1,\mathbb{H}^L)$ in terms of generators of the group as follows

$$u = e^{(i\theta_x + j\theta_y + k\theta_z)/2}. \tag{56}$$

If $u \in U(1,\mathbb{H}^L)$ is related to an $\mathcal{R} \in SO(3)$, then it is clear that $-u \in U(1,\mathbb{H}^L)$ is related to $\mathcal{R}$ also. Therefore, the relation between $U(1,\mathbb{H}^L)$ and $SO(3)$ is not isomorphic, but homomorphic.

4 CONCLUSION

We showed that $U(1,\mathbb{H}^L)$ is the quaternionic version of rotation group $SO(3)$. The generators are as given in eqs.(54) and satisfying commutation relations given in eqs.(55). The relation of $U(1,\mathbb{H}^L)$ and $SO(3)$ is not isomorphic, but homomorphic.

REFERENCES

[1] De Leo, S, Rodrigues W.A., 1995, Quaternion and Special Relativity, Journal Mathematical Physics, Vol.37, No.6

[2] De Leo, S, Rodrigues W.A., 1999, Quaternion Groups in Physics, International Journal of Theoretical Physics, Vol. 38, No.8

[3] Rahmawati, L., 2007, Skripsi : Tinjauan Grup-Grup Simetri Teori Relativitas Khusus Dalam Aljabar Kuaternion Real Dan Penerapannya Dalam Struktur Persamaan Dirac, FMIPA UGM, Yogyakarta.