Boundedness of Pseudo-Differential Operator for $S^0$ Class

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Abstracts

Pseudo-differential operator in a function space is obtained from the Fourier transform of this function space with a multiplier function. This paper will discuss and prove the boundedness of pseudo-differential operator in Lebesgue space with the multiplier function is in the class $S^0$. The evolution of the pseudo-differential theory was then rapid. Based on development from this history, it has gave birth to the general definition of pseudo-differential operator that explain in this paper The general definition of pseudo-differential, of course has an applied or representation. One of them is the problem of partial differential equations in the Poisson equation have the solution, and using Fourier transforms is obtained. In this case the form can be carried in the general form of a pseudo-differential operator. The solution can be estimated for every if operator is a bounded operator. In this paper, the operator defined with corresponds some symbol that describe this operator is more interest. The conclusion of this paper is the boundedness pseudo-differential operator, so we can estimated this number.

Keywords: Pseudo-differential operator; the class $S^0$; Fourier invers function; Lebesgue space.
1. Introduction

Around 1957, Calderón proved the local uniqueness theorem of the Cauchy problem of a partial differential equation [16]. This proof involved the idea of studying the algebraic theory of characteristic polynomials of differential equations [18].

Another landmark was set in ca. 1963, Atiyah and Singer presented their celebrated index theorem. Applying operators, which nowadays are recognised as pseudo-differential operators, it was shown that the geometric and analytical indices of Fredholm operator on a compact manifold are equal. In particular, these successes by Calderón and Atiyah-Singer motivated developing a comprehensive theory for these newly found tools [18]. The Atiyah Singer index theorem is also tied to the advent of K-theory, a significant field of study in itself [24].

The evolution of the pseudo-differential theory was then rapid [15]. Based on development from this history, it has gave birth to the general definition of pseudo-differential operator that expand in this paper. The general definition of pseudo-differential, of course has an applied or representation. One of them is the problem of partial differential equations in the Poisson equation \(\Delta u = f\) which have the solution of the Equation \(|\xi|^2 \hat{u} = \hat{f}\), and using Fourier transforms is obtained

\[
u(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix \cdot \xi} \hat{u}(\xi) d\xi = (2\pi)^{-\frac{n}{2}} \int e^{ix \cdot \xi} \frac{1}{|\xi|^2} \hat{f}(\xi) d\xi[19]\]

In this case the \(u\) form can be carried in the general form of a pseudo-differential operator [24]. The \(u\) solution can be estimated for every \(x\) if operator \(u\) is a bounded operator [1].

2. Study literature

Pseudo-Differential Operator

In this paper, the operator \(T_\sigma\) corresponds to the \(\sigma\) symbol that defined as

\[
T_\sigma(\varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi,
\]

\(\varphi \in L^p(\mathbb{R}^n)\)

when \(\varphi \in L^p(\mathbb{R}^n), \sigma(x, \xi) \in S^k, \) and \(\hat{\varphi}\) Fourier transform of \(\varphi\) [24]. Then the definition \(S^k\) is a set of \(\sigma(x, \xi)\) function in \(C^\infty(\mathbb{R}^n \times \mathbb{R}^n)\) [7] such that for all multi-index \(\alpha\) and \(\beta\), there is exist a positive constant \(C_{\alpha, \beta}\) that only depend \(\alpha\) and \(\beta\), so that

\[
|(D_\xi^\alpha D_\eta^\beta \sigma)(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{k - |\beta|},
\]

\(x, \xi \in \mathbb{R}^n\)

for \(k \in (-\infty, \infty)[24]\). The boundedness \(T_\sigma\) in \(L^p(\mathbb{R}^n)\) is an extension of the boundedness \(T_\sigma\) in \(L^p(\mathbb{R}^n)\) in Schwartz space \((S)\) [2].

**Definition 2.1** Schwartz space \((S)\) is the set of all infinitely partial differentiable \(\phi\) functions on \(\mathbb{R}^n\) such that for all multi-index \(\alpha\) and \(\beta\),

\[
\sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta \phi)(x)| < \infty
\]

Based on Definition 2.1, it’s clear that \(C_0^\infty(\mathbb{R}^n) \subseteq S[14]\). Therefore, there is a theorem saying that

**Theorem 2.1** \(C_0^\infty(\mathbb{R}^n)\) dense in \(L^p(\mathbb{R}^n)\) for \(1 \leq p < \infty\) [6].

This will lead to the corollary \(S\) dense \(L^p(\mathbb{R}^n)\) for \(1 \leq p < \infty\). The density \(S\) in \(L^p(\mathbb{R}^n)\) for \(1 \leq p < \infty\) was shown the \(T_\sigma\) in Schwartz space \((S)\) can be extended in \(L^p(\mathbb{R}^n)\) [1].

**Find and prove new theorem about Pseudo-Differential Operator**

Boundedness of pseudo-differential operators have been shown in

Let \(\sigma\) symbol in \(S^0\), then operator \(T_\sigma: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)\) for \(1 \leq p < \infty\) is a bounded linear operator, with
\((T_{\sigma}\varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi \) [24].

We must remember that Fourier Transform \(\mathcal{F}\) is mapping \(\mathcal{S}\) continuously to \(\mathcal{S}\). Exactly, If \(\varphi_k \rightarrow 0\) in \(\mathcal{S}\) when \(k \rightarrow \infty\), so \(\hat{\varphi}_k \rightarrow 0\) when \(k \rightarrow \infty\) [5].

**Proof:**

Suppose \(\alpha\) and \(\beta\) multi-indeks. So,

\[
|\xi^\alpha (D^\beta \varphi_k)(\xi)| = |(D^\alpha ((-x)^\beta \varphi_k))(\xi)|
\]

\[
= 2\pi^{-\frac{n}{2}} \int e^{-ix \cdot \xi} |(D^\alpha ((-x)^\beta \varphi_k))(x)| dx
\]

\[
\leq 2\pi^{-\frac{n}{2}} \int |e^{-ix \cdot \xi}| |(D^\alpha ((-x)^\beta \varphi_k))(x)| dx
\]

\[
= 2\pi^{-\frac{n}{2}} \int |(D^\alpha ((-x)^\beta \varphi_k))(x)| dx
\]

\[
= (2\pi)^{-\frac{n}{2}} \parallel D^\alpha ((-x)^\beta \varphi_k) \parallel_1, \quad \xi \in \mathbb{R}^n
\]

Because \(\varphi_k \rightarrow 0\) in \(\mathcal{S}\) then \(D^\alpha ((-x)^\beta \varphi_k) \rightarrow 0\) in \(\mathcal{S}\) when \(k \rightarrow \infty\). So we have \(\parallel D^\alpha ((\sup_{\xi \in \mathbb{R}^n} |\xi^\alpha (D^\beta \varphi_k)(\xi)| \rightarrow 0 - x)^\beta \varphi_k) \parallel_1 \rightarrow 0\) in \(k \rightarrow \infty\). This meaning is when \(k \rightarrow \infty\). Its prove that \(\hat{\varphi}_k \rightarrow 0\) in \(\mathcal{S}\) when \(k \rightarrow \infty\) [4].

And also We have \(T_{\sigma}\) mapping \(\mathcal{S}\) continuously to \(\mathcal{S}\). This meaning if \(\varphi_k \rightarrow 0\) in \(\mathcal{S}\), then \(T_{\sigma} \varphi_k \rightarrow 0\) when \(k \rightarrow \infty\) [3].

**Proof:**

Suppose \(\sigma \in \mathcal{S}'^m\), and we have multi-indeks \(\alpha\) and \(\beta\), and positive \(C_{\alpha, \beta, \gamma, \delta}\) only depend \(\alpha, \beta, \gamma, \) and \(\delta\), then
3. Result and Discussion

Boundedness of Pseudo-Differential Operator for $S^k$ Class

Boundedness of pseudo-differential operators have been shown in

**Theorem 3.1** Let $\sigma$ symbol in $S^0$, then operator $T_\sigma: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $1 < p < \infty$ is a bounded linear operator, with

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi.$$ [24].

From this Theorem 3.1, this leads to a more general theorem, that is

**Theorem 3.2** Let $\sigma$ symbol in $S^{-k}$ where $0 \leq k < n$, then operator $T_\sigma: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $1 < p, q < \infty$ is a bounded linear operator, with

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi.$$ when $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$

Before proving the Theorem 3.2, a theorem is needed that is Young Inequality Theorem.

**Theorem 3.3 (Young Inequality)**

Let $1 < p, q, r < \infty$ satisfy

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p} + \frac{1}{r},$$

then there exist a constant $B > 0$ such that for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, its have

$$\| f * g \|_{L^q(\mathbb{R}^n)} \leq B \| f \|_{L^r,\infty(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}.$$ [4].

Proof of Theorem 3.2

Let $\phi(x, \xi) = |\xi|^k \sigma(x, \xi)$, from the characteristics of the product between $S^k$ classes, $\phi$ is symbol in $S^0$. Then, based on Theorem 3.1

$$(T_\phi \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \phi(x, \xi) \hat{\varphi}(\xi) d\xi$$
is bounded linear operator, and

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{1}{|\xi|^k} \phi(x, \xi) \hat{\varphi}(\xi) d\xi.$$ Let $\frac{1}{|\xi|^k} = \hat{f}(\xi)$, and $\hat{\varphi}(\xi) = \phi(x, \xi) \hat{\varphi}(\xi)$. Based on the inversion Fourier and convolution properties, then $T_\sigma = F^{-1}(\hat{f} \hat{\varphi})$. So,

$$\| T_\sigma \|_{L^q(\mathbb{R}^n)} = \| F^{-1}(\hat{f} \hat{\varphi}) \|_{L^q(\mathbb{R}^n)}$$

$$= \| f * g \|_{L^q(\mathbb{R}^n)}$$

Based on Young inequality, it follow that

$$\| T_\sigma \|_{L^q(\mathbb{R}^n)} \leq B \| f \|_{L^r,\infty(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}$$

According to the fact that $\frac{1}{|\xi|^k} = \hat{f}(\xi)$, and $\hat{\varphi}(\xi) = \phi(x, \xi) \hat{\varphi}(\xi)$, it obvious that $f(x) = \frac{1}{|x|^{n-k}}$ and $g(x) = (T_\phi \varphi)(x)$. So that $\frac{1}{|x|^{n-k}} \in L_r,\infty(\mathbb{R}^n)$ if $q = n/(n-k)$, and $\| g \|_{L^p(\mathbb{R}^n)} \leq C \| \phi \|_{L^p(\mathbb{R}^n)}$. Then, the Young inequality requires

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$$

so that,

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$$

$$= \frac{1}{p} + \frac{n-k}{n} - 1$$

$$= \frac{1}{p} + 1 - \frac{k}{n}$$

$$= \frac{1}{p} - \frac{k}{n}$$
\[
\| T_\sigma \|_{L^q(\mathbb{R}^n)} \leq B \| f \|_{L^{p,\infty}(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)} \\
\leq B \| f \|_{L^{p,\infty}(\mathbb{R}^n)} C \| g \|_{L^p(\mathbb{R}^n)}
\]
\[
\varphi \|_{L^p(\mathbb{R}^n)} \leq D \| \varphi \|_{L^p(\mathbb{R}^n)}, \quad D > 0
\]
when \( \frac{1}{q} = \frac{1}{p} - \frac{k}{n} \)

4. Conclusion

The conclusion of this paper is the boundedness pseudo-differential operator in the Theorem 3.2. Let \( \sigma \) symbol in \( S^{-k} \) where \( 0 \leq k < n \), then operator \( T_\sigma : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \) for \( 1 < p, q < \infty \) is a bounded linear operator, with

\[
(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi.
\]
when \( \frac{1}{q} = \frac{1}{p} - \frac{k}{n} \)

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References


