Abstract

The aim of this paper is to provide a complete purely algebraic proof of homomorphism between $SU(2)$ and $SO(3)$ without concerning the topology of both groups. The proof is started by introducing a map $\varphi : SU(2) \rightarrow ML(3, \mathbb{C})$ defined as $\varphi(U)_{ij} = \frac{1}{2} \text{tr}(\sigma_i U \sigma_j U^\dagger)$. Firstly we proof that the map $\varphi$ satisfies $\varphi(U_1 U_2)_{ij} = [\varphi(U_1)]_k^i [\varphi(U_2)]_k^j$, for every $U_1, U_2 \in SU(2)$. The next step is to show that the collection of $\varphi(U)$ is having orthogonal property and every $\varphi(U)$ has determinant of 1. After that, we proof that $\varphi(I_2) = I_3$. Finally, to make sure that $\varphi$ is indeed a homomorphism, not an isomorphism, we proof that $\varphi(-U) = \varphi(U)$, $\forall U \in SU(2)$.

1. Introduction

The study of rotation groups has been carried out in both mathematics and physics and in various topics. Discrete rotation has been studied in [14, 16]. Topologically and geometrically, rotation group has been studied in [17, 22, 8, 20, 18, 7, 19]. In representation theory, rotation groups has been studied in [2, 16, 1, 4, 15]. In physics, the application of rotation group has been discussed in [24, 1, 11, 10, 25].

Any transformation in a vector space are classified as rotation transformation if it preserve the norm of vectors in that vector space. Some authors called it as orthogonal transformations [12]. The set of rotation transformations in a vector space usually form a group classified as rotation group.

As an example, if we let the vector space is $\mathbb{R}^3$, then the rotation group in that space is $SO(3)$, which is defined by [24]

$$SO(3) \equiv \{ A \in GL(3, \mathbb{R}) | AA^T = A^T A = I_3 \}.$$  \hspace{1cm} (1)

As another example, if we let the vector space is $\mathbb{H}$, a space of all $2 \times 2$ complex hermitian traceless matrices, that is [24]

$$\mathbb{H} \equiv \{ H \in ML(2, \mathbb{C}) | H^\dagger = H \ \text{dan} \ \text{tr}(H) = 0 \},$$  \hspace{1cm} (2)

then the rotation group is $SU(2)$ which is defined by [24]

$$SU(2) \equiv \{ U \in GL(2, \mathbb{C}) | U^\dagger U = UU^\dagger = I_2 \ \text{and} \ \text{det}(U) = 1 \}.$$  \hspace{1cm} (3)

We also know that one of the topological properties of $SU(2)$ is simply connectedness [9]. Meanwhile, $SO(3)$ is not a simply connected topological group[23].
In various literatures discussing groups $SU(2)$ and $SO(3)$, we usually find a statement that there is a homomorphism from $SU(2)$ to $SO(3)$. The homomorphism of group $SU(2)$ to $SO(3)$ play an important role in quantum mechanics, especially when we dealing with electron spin of Pauli theory [3, 13, 21]. Cornwell in [5] give the proof by considering the simply connectedness of $SU(2)$, especially when arguing that the homomorphism maps any elements of $SU(2)$ into $SO(3)$, rather than into $O(3)$. Donchev et.al in [6] gave the proof by using the Cayley maps for the isomorphic Lie algebras $su(2)$ and $so(3)$. Sattinger and Weaver in [23] construct the homomorphism between $su(3)$ and $so(3)$. Dealing with electron spin of Pauli theory [3, 13, 21].

Nevertheless, as long as our searching in various literatures, we never found a complete explicit computation of homomorphism from $SU(2)$ to $SO(3)$ by purely algebraic ways, without concerning their topological properties. Motivated by this fact, in this paper we give an explicit completely purely algebraic proof of homomorphism of $SU(2)$ to $SO(3)$ without concerning their topological properties. We hope this research will give an alternative explanation of homomorphism $SU(2)$ to $SO(3)$ without having to learn topology first.

2. Rotation in $\mathbb{R}^3$

Each element $X$ in $\mathbb{R}^3$ can be expressed in the following form

$$X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where $x^1, x^2, x^3 \in \mathbb{R}$. The norm of $X$ is defined by

$$|X|^2 = X^T X = (x^1)^2 + (x^2)^2 + (x^3)^2.$$  (4)

Every $A \in SO(3)$ is a rotation transformation in $\mathbb{R}^3$ since if $X' = AX$, where $X \in \mathbb{R}^3$, then it follow that

$$|X'|^2 = X'^T X' = (AX)^T (AX) = X^T A^T AX = X^T \mathbb{I}_3 X = X^T X = |X|^2.$$  (5)

3. Rotation in $\mathbb{H}$

According to the definition of $\mathbb{H}$, every $V \in \mathbb{H}$ may be expressed in the following form

$$V = \begin{pmatrix} x^1 \\ x^2 + ix^2 \\ x^3 - ix^2 \\ -x^3 \end{pmatrix}.$$  (7)

One of bases in the vector space $\mathbb{H}$ are the following three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that each vector $V \in \mathbb{H}$ may be expressed as follow

$$V = x^1 \sigma_1 + x^2 \sigma_2 + x^3 \sigma_3,$$  (9)

Note that the above three Pauli matrices satisfy the following properties

$$tr(\sigma_i) = 0,$$  (10)

$$\sigma_i^\dagger = \sigma_i,$$  (11)

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{I}_2 + \delta_{ij} \mathbb{I}_2,$$  (12)

for all $i, j, k = 1, 2, 3$, where $\epsilon_{ijk}$ is defined by

$$\epsilon_{ijk} = \begin{cases} 1, & (ijk) = (123), (231), (312) \\ -1, & (ijk) = (132), (321) \\ 0, & \text{others} \end{cases}$$  (13)

and $\delta_{ij}$ is a kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$  (14)

The norm of $V \in \mathbb{H}$ is defined by

$$|V|^2 = -\det(V) = (x^1)^2 + (x^2)^2 + (x^3)^2.$$  (15)

The rotation of $V$ by $U \in SU(2)$ is defined by

$$V' = UVU^\dagger,$$  (16)

since $V'$ is hermitian matrix, that is

$$V'^\dagger = (UVU^\dagger)^\dagger = (U^\dagger)^\dagger V^\dagger U^\dagger = UVU^\dagger = V',$$  (17)

and traceless, that is

$$tr(V') = tr(UVU^\dagger) = tr(VU^\dagger U) = tr(V\mathbb{I}_2) = tr(V) = 0,$$  (18)

and also having same norm as $V$, that is

$$|V'|^2 = -\det(V') = -\det(UVU^\dagger) = -\det(U)\det(V)\det(U^\dagger) = -\det(V) \cdot 1 = -\det(V) = |V|^2.$$  (19)

4. Homomorphism from $SU(2)$ to $SO(3)$

In order to find a homomorphism from $SU(2)$ to $SO(3)$, we note that $V' = x^\dagger \sigma_i = UVU^\dagger$ and $V = x^\dagger \sigma_i$. Hence, by using eq.(12) we obtain

$$x = \frac{1}{2} tr(\sigma_i V) = \frac{1}{2} tr(\sigma_i UVU^\dagger) = \frac{1}{2} tr(\sigma_i U x^\dagger \sigma_i U^\dagger).$$  (20)

$$x = \frac{1}{2} tr(\sigma_i U x^\dagger \sigma_i U^\dagger) x^\dagger.$$
Meanwhile we know that if a vector \( X = (x^1, x^2, x^3)^T \) is transformed by a matrix \( A \in O(3) \), then we will get a new vector, say \( V^\prime = (x'^1, x'^2, x'^3)^T \), according to formula
\[
x'^i = [A]^i_j x^j. \tag{21}
\]
Hence we may conclude that the entries of matrix \( A \in SO(3) \) may be written in the expression of the Pauli matrices as follows
\[
[A]^i_j = \frac{1}{2} \text{tr}(\sigma_i U \sigma_j U^\dagger). \tag{22}
\]
More over, we can try to start from a map \( \varphi : SU(2) \rightarrow ML(3, \mathbb{C}) \) defined by
\[
[\varphi(U)]^i_j = \frac{1}{2} \text{tr}(\sigma_i U \sigma_j U^\dagger), \tag{23}
\]
and then show that \( [\varphi(U)] \) is in \( SO(3) \) whenever \( U \in SU(2) \). According to eq.22, eq.21 and eq.20, it is clear that the map \( \varphi(U) \) defined in eq.23 is belong to \( O(3) \). However in this article we will show that \( \varphi(U) \) in eq.(23) is an element of \( O(3) \) by using the property of orthogonality of the elements of \( O(3) \), i.e. for every \( A \in O(3) \) we have
\[
A A^T = T^A A = I_3. \tag{24}
\]
Now for first calculation we will prove that the map \( \varphi \) satisfy
\[
[\varphi(U_1 U_2)]^i_j = [\varphi(U_1)]^i_k [\varphi(U_2)]^k_j, \tag{25}
\]
for every \( U_1, U_2 \in SU(2) \). This will provide us the homomorphism property of the maps defined in eq.(23). By using eq.(23), the right side of eq.(25) become
\[
[\varphi(U_1)]^i_k [\varphi(U_2)]^k_j = \frac{1}{4} \text{tr}(\sigma_i U_1 \sigma_k U_2) \frac{1}{2} \text{tr}(\sigma_k U_2 \sigma_j U_1) = \frac{1}{4} \text{tr}(\sigma_i U_1 \sigma_k U_2) \text{tr}(\sigma_k U_2 \sigma_j U_1) = \frac{1}{4} \text{tr}(\sigma_i \Omega_{k1}) \text{tr}(\sigma_k \Omega_{j2}), \tag{26}
\]
where
\[
\Omega_{k1} \equiv U_1 \sigma_k U_1, \quad \Omega_{j2} \equiv U_2 \sigma_j U_2. \tag{27}
\]
According to the definition of trace, eq.(26) become
\[
[\varphi(U_1)]^i_k [\varphi(U_2)]^k_j = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}]^2 [\Omega_{j2}]) = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]) = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]), \tag{28}
\]
where
\[
\Xi_{\alpha \beta \gamma} = [\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]. \tag{29}
\]
Note that since \( \alpha, \beta, \gamma, \delta = 1, 2 \) there are 16 combinations of \( \alpha, \beta, \gamma, \delta \). The computation of those 16 values of \( \Xi_{\alpha \beta \gamma} \) are given below :
\[
\Xi_{1111} = [\sigma_1]^2 [\sigma_1]^2, \quad \Xi_{1122} = [\sigma_1]^2 [\sigma_2]^2, \quad \Xi_{1212} = [\sigma_2]^2 [\sigma_1]^2, \quad \Xi_{2212} = [\sigma_2]^2 [\sigma_2]^2,
\]
and so on. Now using the above 16 values of \( \Xi_{\alpha \beta \gamma} \), eq.(28) become
\[
[\varphi(U_1)]^i_k [\varphi(U_2)]^k_j = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]) = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]) = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]) = \frac{1}{4} \text{tr}([\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]),
\]
where
\[
\Xi_{\alpha \beta \gamma} = [\sigma_k]^2 [\Omega_{k1}, \Omega_{j2}]. \tag{29}
\]
are hermitian matrices, because

Finally, according to the de-

get

Next we will prove the following two conditions

for all $U \in SU(2)$. The first condition is needed to ensure that the collections of $\varphi(U)$ is having orthogonal property and the second condition is needed to ensure that $\varphi(U)$ is belong to $SL(3, \mathbb{R})$, for every $U \in SU(2)$. The first condition may be written in the expression of matrix entries as follows

Since

then the left side of eq.(36) become

By doing the similar computation as was done from eq.(28) until eq.(33), then the last equation become

Finally, according to the definition of $\Omega_{ki}$ and $\Omega'_{ki}$, we
However according to eq.(10) and $tr(I_2) = 2$, we obtain
\[ [\varphi(U)]^I_\alpha^I_\beta \] (41)
that is $\varphi(U) \in O(3), \forall U \in SU(2)$.

For the second condition in eq.(35), according to the definition of determinant of a matrix, $det(\varphi(U))$ may be written in the following form
\[ det(\varphi(U)) = e^{ijk} [\varphi(U)]^I_\alpha^I_\beta [\varphi(U)]^I_\gamma^I_k \] (42)

Using the definition of $\varphi(U)$, then we have
\[ det(\varphi(U)) = e^{ijk} \frac{1}{2} tr(\sigma_3 U \sigma_3 U^\dagger) \frac{1}{2} tr(\sigma_3 U \sigma_3 U^\dagger) \]
\[ = \frac{1}{8} e^{ijk} tr(\sigma_3 U \sigma_3 U^\dagger) \frac{1}{2} tr(\sigma_3 U \sigma_3 U^\dagger) \]
\[ = \frac{1}{8} e^{ijk} \delta_\delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta \δ
1, 2, 3, are given below

\[
\begin{align*}
[\Omega_1^1]_2 &= \lbrack U_1 \sigma_1 U_1^\dagger \rbrack_2^2 \\
&= (U_1)^2_2 (\sigma_1)^2_2 (U_1^\dagger)^2_2 + (U_1)^1_2 (\sigma_1)^2_1 (U_1^\dagger)^1_2 \\
&= \cos \theta e^{i\xi} \cos \theta e^{i\nu} - \sin \theta e^{i\eta} \sin \theta e^{i\eta} \\
&= \cos^2 \theta e^{i2\xi} - \sin^2 \theta e^{i2\eta} \\
\end{align*}
\]

(49)

\[
\begin{align*}
[\Omega_2^1]_1 &= \lbrack U_2 \sigma_1 U_2^\dagger \rbrack_1^2 \\
&= (U_2)^2_1 (\sigma_1)^2_2 (U_2^\dagger)^2_1 + (U_2)^1_1 (\sigma_1)^2_1 (U_2^\dagger)^1_1 \\
&= \sin \theta e^{i\eta} (-\sin \theta e^{i\eta}) + \cos \theta e^{-i\xi} \cos \theta e^{-i\n} \\
&= -\sin^2 \theta e^{-i2\eta} + \cos^2 \theta e^{-i2\xi} \\
\end{align*}
\]

(50)

\[
\begin{align*}
[\Omega_1^1]_1 &= \lbrack U_1 \sigma_1 U_1^\dagger \rbrack_1^1 \\
&= (U_1)^2_1 (\sigma_1)^2_1 (U_1^\dagger)^2_1 + (U_1)^1_1 (\sigma_1)^2_1 (U_1^\dagger)^1_2 \\
&= -\sin \theta e^{i\eta} \cos \theta e^{-i\xi} - \cos \theta e^{i\eta} \sin \theta e^{-i\nu} \\
&= -\cos \theta \sin \theta e^{-i(\zeta - \eta)} + e^{i(\zeta - \eta)} \\
&= -2 \cos \theta \sin \theta \cos(\zeta - \eta) \\
\end{align*}
\]

(51)

\[
\begin{align*}
[\Omega_2^1]_2 &= \lbrack U_2 \sigma_1 U_2^\dagger \rbrack_2^1 \\
&= (U_2)^2_2 (\sigma_1)^2_1 (U_2^\dagger)^2_1 + (U_2)^1_1 (\sigma_1)^2_1 (U_2^\dagger)^1_1 \\
&= -\sin \theta e^{i\eta}(i) \sin \theta e^{i\eta} + \cos \theta e^{i\xi} (-i) \cos \theta e^{i\nu} \\
&= -i(\sin^2 \theta e^{2i\eta} + \cos^2 \theta e^{2i\xi}) \\
\end{align*}
\]

(52)

\[
\begin{align*}
[\Omega_2^1]_1 &= \lbrack U_2 \sigma_2 U_2^\dagger \rbrack_1^1 \\
&= (U_2)^2_1 (\sigma_2)^2_2 (U_2^\dagger)^2_1 + (U_2)^1_1 (\sigma_2)^2_1 (U_2^\dagger)^1_1 \\
&= \sin \theta e^{-i\eta} (-i) - \sin \theta e^{-i\eta} \\
&+ \cos \theta e^{-i\xi} \cos \theta e^{-i\nu} \\
&= i(\sin^2 \theta e^{-2i\eta} + \cos^2 \theta e^{-2i\xi}) \\
\end{align*}
\]

(53)

\[
\begin{align*}
[\Omega_2^1]_2 &= \lbrack U_2 \sigma_2 U_2^\dagger \rbrack_2^2 \\
&= (U_2)^2_2 (\sigma_2)^2_2 (U_2^\dagger)^2_2 + (U_2)^1_1 (\sigma_2)^2_1 (U_2^\dagger)^1_1 \\
&= \cos \theta e^{i\xi} (-i) (-\sin \theta e^{i\eta}) \\
&- \sin \theta e^{i\eta}(i) \cos \theta e^{i\xi} \\
&= i \cos \theta \sin \theta e^{i(\zeta - \eta)} - e^{-i(\zeta - \eta)} \\
&= i \cos \theta \sin \theta (2i \sin(\zeta - \eta)) \\
&= -2 \cos \theta \sin \theta \sin(\zeta - \eta) \\
\end{align*}
\]

(54)

\[
\begin{align*}
[\Omega_3^1]_1 &= \lbrack U_3 \sigma_1 U_3^\dagger \rbrack_1^1 \\
&= (U_3)^2_1 (\sigma_3)^2_1 (U_3^\dagger)^2_1 + (U_3)^1_1 (\sigma_3)^2_1 (U_3^\dagger)^1_2 \\
&= \cos \theta e^{i\xi} \cos \theta e^{i\nu} + \sin \theta e^{i\eta} \sin \theta e^{i\eta} \\
&= 2 \cos \theta \sin \theta e^{i(\zeta + \eta)} \\
\end{align*}
\]

(55)

\[
\begin{align*}
[\Omega_3^1]_2 &= \lbrack U_3 \sigma_3 U_3^\dagger \rbrack_2^2 \\
&= (U_3)^2_2 (\sigma_3)^2_2 (U_3^\dagger)^2_2 + (U_3)^1_1 (\sigma_3)^2_1 (U_3^\dagger)^1_1 \\
&= \sin \theta e^{i\xi} \sin \theta e^{i\eta} + \cos \theta e^{i\nu} \sin \theta e^{i\eta} \\
&= 2 \cos \theta \sin \theta e^{i(\zeta + \eta)} \\
\end{align*}
\]

(56)

\[
\begin{align*}
[\Omega_3^1]_1 &= \lbrack U_3 \sigma_3 U_3^\dagger \rbrack_1^1 \\
&= (U_3)^1_1 (\sigma_3)^1_1 (U_3^\dagger)^1_1 + (U_3)^1_1 (\sigma_3)^2_1 (U_3^\dagger)^1_2 \\
&= \cos \theta e^{i\xi} \cos \theta e^{i\nu} + (-\sin \theta e^{i\eta})(-1)(-\sin \theta e^{i\eta}) \\
&= \cos^2 \theta - \sin^2 \theta \\
\end{align*}
\]

(57)

Now, we can compute the values of \(\text{Im}(\lbrack \Omega_i^1 \rbrack_2^2 \lbrack \Omega_j^1 \rbrack_1^2),\) for all \(i, j = 1, 2, 3,\) as follows

\[
\begin{align*}
\text{Im}(\lbrack \Omega_1^1 \rbrack_2^2 \lbrack \Omega_2^1 \rbrack_1^2) &= \text{Im}(\cos^2 \theta e^{2i\xi} - \sin^2 \theta e^{2i\eta}) \\
&\times (i \cos^2 \theta e^{-2i\xi} + \sin^2 \theta e^{-2i\eta})) \\
&= \text{Im}(i \cos^4 \theta - \sin^4 \theta) \\
&+ \cos^2 \theta \sin^2 \theta e^{4i(\zeta - \eta)} \\
&- \cos^2 \theta \sin^2 \theta e^{-4i(\zeta - \eta)}) \\
&= \text{Im}(i \cos^4 \theta - \sin^4 \theta) \\
&+ \cos^2 \theta \sin^2 \theta (\cos 2(\zeta - \eta)) \\
&+ \sin 2(\zeta - \eta))) \\
&- \cos^2 \theta \sin^2 \theta (\cos 2(\zeta - \eta)) \\
&- \sin 2(\zeta - \eta)) \\
&= \cos^4 \theta - \sin^4 \theta \\
\end{align*}
\]

(58)
\begin{align}
\text{Im}[\Omega_{21}^{\dagger}\Omega_{11}^{\dagger}] & = \text{Im}\left((-i \sin^2 \theta e^{2i\eta} - i \cos^2 \theta e^{2i\zeta}\right) \\
& \times (- \sin^2 \theta e^{-2i\eta} + \cos^2 \theta e^{-2i\zeta}) \\
& = \text{Im}(i \sin^4 \theta - \cos^4 \theta \\
& + \cos^2 \theta \sin^2 \theta e^{2i(\zeta - \eta)} \\
& - \cos^2 \theta \sin^2 \theta e^{-2i(\zeta - \eta)}) \\
& = \text{Im}(i \sin^4 \theta - \cos^4 \theta \\
& + \cos^2 \theta \sin^2 \theta (\cos 2(\zeta - \eta) \\
& + i \sin 2(\zeta - \eta)) \\
& - \cos^2 \theta \sin^2 \theta (\cos 2(\zeta - \eta) \\
& - i \sin 2(\zeta - \eta))) \\
& = \text{Im}(i \sin^4 \theta - \cos^4 \theta \\
& + 2 \cos^2 \theta \sin^2 \theta \sin 2(\zeta - \eta))) \\
& = \sin^4 \theta - \cos^4 \theta \\
& \quad \quad (59)
\end{align}

\begin{align}
\text{Im}[\Omega_{31}^{\dagger}\Omega_{11}^{\dagger}] & = \text{Im}(2 \cos \theta e^{i\zeta} \sin \theta e^{i\eta}) \\
& \times (- \sin^2 \theta e^{-2i\eta} + \cos^2 \theta e^{-2i\zeta}) \\
& = 2 \text{Im}(\cos^3 \theta \sin \theta e^{-i(\zeta - \eta)} \\
& - \sin^3 \theta \cos \theta e^{i(\zeta - \eta)}) \\
& = 2 \text{Im}(\cos^3 \theta \sin \theta (\cos(\zeta - \eta) \\
& - \sin(\zeta - \eta)) \\
& - \sin^3 \theta \cos \theta (\cos(\zeta - \eta) \\
& + \sin(\zeta - \eta))) \\
& = -2(\cos^3 \theta \sin \theta + \sin^3 \theta \cos \theta) \\
& \times \sin(\zeta - \eta) \\
& \quad \quad (62)
\end{align}

\begin{align}
\text{Im}[\Omega_{21}^{\dagger}\Omega_{22}^{\dagger}] & = \text{Im}(\cos^2 \theta e^{2i\zeta} - \sin^2 \theta e^{2i\eta}) \\
& \times (2 \cos \theta e^{-i\zeta} \sin \theta e^{-i\eta}) \\
& = 2 \text{Im}(\cos^3 \theta \sin \theta e^{i(\zeta - \eta)} \\
& - \sin^3 \theta \cos \theta e^{-i(\zeta - \eta)}) \\
& = 2 \text{Im}(\cos^3 \theta \sin \theta (\cos(\zeta - \eta) \\
& + \sin(\zeta - \eta)) \\
& - \sin^3 \theta \cos \theta (\cos(\zeta - \eta) \\
& - \sin(\zeta - \eta))) \\
& = (\cos^3 \theta \sin \theta + \sin^3 \theta \cos \theta) \\
& \times \sin(\zeta - \eta) \\
& \quad \quad (60)
\end{align}

\begin{align}
\text{Im}[\Omega_{31}^{\dagger}\Omega_{22}^{\dagger}] & = \text{Im}(2 \cos \theta e^{i\zeta} \sin \theta e^{i\eta}) \\
& \times (i \sin^2 \theta e^{-2i\eta} + i \cos^2 \theta e^{-2i\zeta}) \\
& = 2 \text{Im}(i \sin^3 \theta \cos \theta e^{i(\zeta - \eta)} \\
& + \cos^3 \theta \sin \theta e^{-i(\zeta - \eta)}) \\
& = 2 \text{Im}(i \sin^3 \theta \cos \theta (\cos(\zeta - \eta) \\
& + \sin(\zeta - \eta)) \\
& + \cos^3 \theta \sin \theta (\cos(\zeta - \eta) \\
& - i \sin(\zeta - \eta))) \\
& = 2(\sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta) \\
& \times \cos(\zeta - \eta) \\
& \quad \quad (63)
\end{align}

Finally, eq.(47) become

\begin{align}
det(\varphi(U)) = \frac{1}{2} \\
& \left[ (\cos^4 \theta - \sin^4 \theta) \\
& - (\cos^4 \theta - \sin^4 \theta)(\cos^2 \theta - \sin^2 \theta) \\
& + (2(\sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta) \\
& \times \cos(\zeta - \eta)) \\
& - 2(\sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta) \\
& \right] \\
& \quad \quad (64)
\end{align}
\[
\times \cos(\zeta - \eta)(-2 \cos \theta \sin \theta \cos(\zeta - \eta)) + ((-2 \sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta) \\
\sin(\zeta - \eta)) - 2(\sin^4 \theta \cos \theta) \\
+ \cos^3 \theta \sin \theta) \sin(\zeta - \eta)) \\
\times (-2 \cos \theta \sin \theta \sin(\zeta - \eta)) \\
+ 8(\sin \theta \cos \theta)(\sin^2 \theta + \cos^2 \theta) \\
\times (\cos \theta \sin \theta) \]
\]
(65)

Of course we have
\[
[\varphi(I_3)]_k^* = \frac{1}{2}tr(\sigma_i I_2 \sigma_j I_2^*) = \frac{1}{2}tr(\sigma_i \sigma_j)
\]
\[
= \frac{1}{2}tr(i \varepsilon_{ijk} \sigma_k + \delta_{ij} I_2)
\]
\[
= \frac{1}{2}(i \varepsilon_{ijk}tr(\sigma_k) + \delta_{ij}tr(I_2)
\]
\[
= \frac{1}{2}(0 + 2\delta_{ij}) = \delta_{ij},
\]
so we can conclude that \(\varphi(I_2) = I_3\).

These result shows us \(\varphi(U)\) is in \(SO(3)\) for every \(U\) in \(SU(2)\). Finally by using the result obtained in eq.(34), we concluded that map \(\varphi\) defined in eq.(23) is a homomorphism of \(SU(2)\) to \(SO(3)\). So, instead of considering the topological properties as in [5], we have proved by purely algebraically that the maps defined in eq.(23) will maps any elements of \(SU(2)\) into \(SO(3)\). Moreover, according to definition (23), it follows that
\[
[\varphi(-U)]_j^* = \frac{1}{2}tr(\sigma_i (-U) \sigma_j (-U)^*)
\]
\[
= [\varphi(U)]_j^* \equiv \frac{1}{2}tr(\sigma_i U \sigma_j U^*)
\]
\[
= [\varphi(U)]_j^*,
\]
so we obtain that \(\varphi(-U) = \varphi(U)\), \(\forall U \in SU(2)\).

5. Conclusions

The complete purely algebraic proof of homomorphism between two rotation groups, \(SU(2)\) and \(SO(3)\), was given by introducing a map \(\varphi : SU(2) \rightarrow SO(3)\) defined as
\[
[\varphi(U)]_j^* = \frac{1}{2}tr(\sigma_i U \sigma_j U^*)
\]
The proof was obtained successfully by doing algebraic calculation, without concerning the topology of both groups.

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References


