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Boundedness of Pseudo-Differential Operator for S^0 Class

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Abstracts

Pseudo-differential operator in a function space is obtained from the Fourier transform of this function space with a multiplier function. This paper will discuss and prove the boundedness of pseudo-differential operator in Lebesgue space with the multiplier function is in the class S^0. The evolution of the pseudo-differential theory was then rapid. Based on development from this history, it has gave birth to the general definition of pseudo-differential operator that explain in this paper The general definition of pseudo-differential, of course has an applied or representation. One of them is the problem of partial differential equations in the Poisson equation have the solution , and using Fourier transforms is obtained. In this case the form can be carried in the general form of a pseudo-differential operator. The solution can be estimated for every if operator is a bounded operator. In this paper, the operator defined with correspondes some symbol that describe this operator is more interest. The conclusion of this paper is the boundedness pseudo-differential operator , so we can estimated this number.

Keywords: Pseudo-differential operator; the class S^0; Fourier invers function; Lebesgue space.

1. Introduction

Around1957, Calder´on proved the local uniqueness theorem of the Cauchy problem of a partial differential equation[16]. This proof involved the idea of studying the algebraic theory of characteristic polynomials of differential equations[18].

Another landmark was set in ca.1963, Atiyah and Singer presented their celebrated index theorem. Applying operators, which nowadays are recognised as pseudo-differential operators, it was shown that the geometric and analytical indices of Fredholm operator on a compact manifold are equal. In particular, these successes by Calder´on and Atiyah-Singer motivated developing a comprehensive theory for these newly found tools[18]. The Atiyah Singer index theorem is also tied to the advent of K-theory, a significant field of study in itself[24].

The evolution of the pseudo-differential theory was then rapid[15]. Based on development from this history, it has gave birth to the general def-inition of pseudo-differential operator that explain in this paper The general definition of pseudo-differential, of course has an applied or representation. One of them is the problem of partial differential equations in the Poisson equation $\Delta u = f$ which have the solution of the Equation $|\xi|^2\hat{u} = \hat{f}$, and using Fourier transforms is obtained

$$u(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix.\xi}\, \hat{u} d\xi$$

$$= (2\pi)^{-\frac{n}{2}} \int e^{ix.\xi} \frac{1}{|\xi|^2} \hat{f}(\xi)\, d\xi [19]$$

In this case the u form can be carried in the general form of a pseudo-differential operator[24]. The u solution can be estimated for every x if operator u is a bounded operator[1].

2. Study literature

Pseudo-Differential Operator

In this paper, the operator T_σ corresponds to the σ symbol that defined as

$$T_\sigma(\varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi) d\xi,$$

$$\varphi \in L^p(\mathbb{R}^n)$$

when $\varphi \in L^p(\mathbb{R}^n)$, $\sigma(x,\xi) \in S^k$, and $\hat{\varphi}$ Fourier transform of φ [24]. Then the definition S^k is a set of $\sigma(x,\xi)$ function in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$[7] such that for all multi-index α and β, there is exist a positive constant $C_{\alpha,\beta}$ that only depend α and β, so that

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{k-|\beta|},$$

$$x, \xi \in \mathbb{R}^n$$

for $k \in (-\infty, \infty)$[24]. The boundedness T_σ in $L^p(\mathbb{R}^n)$ is an extension of the boundedness T_σ in $L^p(\mathbb{R}^n)$ in Schwartz space ($\mathcal{S}$)[2].

Definition 2.1 *Schwartz space ($\mathcal{S}$) is the set of all infinitely partial differentiable ϕ functions on $\mathbb{R}^n$ such that for all multi-index α and β,*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta(\phi))(x)| < \infty$$

Based on Definition 2.1, it's clear that $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}$[14]. Therefore, there is a theorem saying that

Theorem 2.1 $C_0^\infty(\mathbb{R}^n)$ *dense in* $L^p(\mathbb{R}^n)$ *for* $1 \leqslant p < \infty$ [6].

This will lead to the corollary $\mathcal{S}$ dense $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$. The density $\mathcal{S}$ in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$ was shown the T_σ in Schwartz space ($\mathcal{S}$) can be extended in $L^p(\mathbb{R}^n)$[1].

Find and prove new theorem about Pseudo-Differential Operator

Boundedness of pseudo-differential operators have been shown in

Let σ symbol in S^0, then operator $T_\sigma: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p < \infty$ is a bounded linear operator, with

$(T_\sigma\varphi)(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)d\xi$ [24].

We must rembember that Fourier Transform $\mathcal{F}$ is mapping $\mathcal{S}$ continously to $\mathcal{S}$. Exactly, If $\varphi_k \to 0$ in $\mathcal{S}$ when $k \to \infty$, so $\hat{\varphi}_k \to 0$ when $k \to \infty$[5].

Proof :

Suppose α and β multi-indeks. So,

$$|\xi^\alpha(D^\beta\hat{\varphi}_k)(\xi)| = |\{D^\alpha((-x)^\beta\varphi_k)\}\hat{}(\xi)|$$

$$= |2\pi^{-\frac{n}{2}}\int e^{-ix\xi}\{D^\alpha((-x)^\beta\varphi_k)\}(x)dx|$$

$$\leq 2\pi^{-\frac{n}{2}}\int |e^{-ix\xi}\{D^\alpha((-x)^\beta\varphi_k)\}(x)|dx$$

$$\leq 2\pi^{-\frac{n}{2}}\int |e^{-ix\xi}||\{D^\alpha((-x)^\beta\varphi_k)\}(x)|dx$$

$$= 2\pi^{-\frac{n}{2}}\int |\{D^\alpha((-x)^\beta\varphi_k)\}(x)|dx$$

$$= (2\pi)^{-\frac{n}{2}}\| D^\alpha((-x)^\beta\varphi_k)\|_1, \quad \xi \in \mathbb{R}^n$$

Because $\varphi_k \to 0$ in $\mathcal{S}$ then $D^\alpha((-x)^\beta\varphi_k) \to 0$ in $\mathcal{S}$ when $k \to \infty$. So we have $\| D^\alpha((\sup_{\xi\in\mathbb{R}^n}|\xi^\alpha(D^\beta\hat{\varphi}_k)(\xi)| \to 0 - x)^\beta\varphi_k)\|_1 \to 0$ in $k \to \infty$. This meaning is

when $k \to \infty$. Its prove that $\hat{\varphi}_k \to 0$ in $\mathcal{S}$ when $k \to \infty$[4].

And also We have T_σ mapping $\mathcal{S}$ continously to $\mathcal{S}$. This meaning if $\varphi_k \to 0$ in $\mathcal{S}$, then $T_\sigma\varphi_k \to 0$ when $k \to \infty$[3].

Proof:

Suppose $\sigma \in S^m$, and we have multi indeks α and β, and positive $C_{\alpha,\beta,\gamma,\delta}$ only depend α,β,γ, and δ, then

$$\sup_{x\in\mathbb{R}^n}|x^\alpha(D^\beta(T_\sigma\varphi_k))(x)|$$

$$= x^\alpha(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} D_x^\beta\{e^{ix\cdot\xi}\sigma(x,\xi)\}\hat{\varphi}_k(\xi)d\xi$$

$$= x^\alpha(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\xi^\gamma e^{ix\cdot\xi}(D_x^{\beta-\gamma}\sigma)(x,\xi)\hat{\varphi}_k(\xi)d\xi$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\xi^\gamma(x^\alpha e^{ix\cdot\xi})(D_x^{\beta-\gamma}\sigma)(x,\xi)\hat{\varphi}_k(\xi)d\xi$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}(D_\xi^\alpha e^{ix\cdot\xi})(D_x^{\beta-\gamma}\sigma)(x,\xi)\xi^\gamma\hat{\varphi}_k(\xi)d\xi$$

$$= (2\pi)^{-\frac{n}{2}}(-1)^{|\alpha|}\int_{\mathbb{R}^n}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}e^{ix\cdot\xi}D_\xi^\alpha\{(D_x^{\beta-\gamma}\sigma)(x,\xi)\xi^\gamma\hat{\varphi}_k(\xi)\}d\xi$$

$$= (2\pi)^{-\frac{n}{2}}(-1)^{|\alpha|}\int_{\mathbb{R}^n}\sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}e^{ix\cdot\xi}(D_\xi^{\alpha-\delta}D_x^{\beta-\gamma}\sigma)(x,\xi)D_\xi^\delta(\xi^\gamma\hat{\varphi}_k(\xi))nd\xi$$

$$\leq (2\pi)^{-\frac{n}{2}}\sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C_{\alpha,\beta,\gamma,\delta}\int_{\mathbb{R}^n}(1+|\xi|)^{m-|\alpha|+|\delta|}|D_\xi^\delta(\xi^\gamma\hat{\varphi}_k(\xi))|d\xi$$

So,

$$\sup_{x\in\mathbb{R}^n}|x^\alpha(D^\beta(T_\sigma\varphi_k))(x)| \leq (2\pi)^{-\frac{n}{2}}\sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C_{\alpha,\beta,\gamma,\delta}\int_{\mathbb{R}^n}(1+|\xi|)^{m-|\alpha|+|\delta|}|D_\xi^\delta(\xi^\gamma\hat{\varphi}_k(\xi))|d\xi.$$ [5]

3. Result and Discussion

Boundedness of Pseudo-Differential Operator for S^k Class

Boundedness of pseudo-differential operators have been shown in

Theorem 3.1 *Let σ symbol in S^0, then operator $T_\sigma: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p < \infty$ is a bounded linear operator, with*

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)d\xi$$

[24].

From this Theorem 3.1, this leads to a more general theorem, that is

Theorem 3.2 *Let σ symbol in S^{-k} where $0 \leqslant k < n$, then operator $T_\sigma: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $1 < p, q < \infty$ is a bounded linear operator, with*

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)d\xi.$$

when $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.

Before proving the Theorem 3.2, a theorem is needed that is Young Inequality Theorem.

Theorem 3.3 (Young Inequality)

Let $1 < p, q, r < \infty$ satisfy

$$\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p},$$

then there exist a constant $B > 0$ such that for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, its have

$$\| f * g \|_{L^q(\mathbb{R}^n)} \leqslant B \| f \|_{L^{r,\infty}(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}$$

[4].

Proof of Theorem 3.2

Let $\phi(x,\xi) = |\xi|^k \sigma(x,\xi)$, from the characteristics of the product between S^k classes, ϕ is symbol in S^0. Then, based on Theorem 3.1

$$(T_\phi \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \phi(x,\xi)\hat{\varphi}(\xi)d\xi$$

is bounded linear operator, and

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)d\xi$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{|\xi|^k} \phi(x,\xi)\hat{\varphi}(\xi)d\xi.$$

Let $\frac{1}{|\xi|^k} = \hat{f}(\xi)$, and $\hat{g}(\xi) = \phi(x,\xi)\hat{\varphi}(\xi)$. Based on the inversion Fourier and convolution properties, then $T_\sigma = \mathcal{F}^{-1}(\hat{f}\hat{g})$. So,

$$\| T_\sigma \|_{L^q(\mathbb{R}^n)} = \| \mathcal{F}^{-1}(\hat{f}\hat{g}) \|_{L^q(\mathbb{R}^n)}$$

$$= \| f * g \|_{L^q(\mathbb{R}^n)}$$

Based on Young inequality, it follow that

$$\| T_\sigma \|_{L^q(\mathbb{R}^n)} \leqslant B \| f \|_{L^{r,\infty}(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}$$

According to the fact that $\frac{1}{|\xi|^k} = \hat{f}(\xi)$, and $\hat{g}(\xi) = \phi(x,\xi)\hat{\varphi}(\xi)$, it obvious that $f(x) = \frac{1}{|x|^{n-k}}$ and $g(x) = (T_\phi \varphi)(x)$. So that $\frac{1}{|x|^{n-k}} \in L^{r,\infty}(\mathbb{R}^n)$ if $q = n/(n-k)$, and $\| g \|_{L^p(\mathbb{R}^n)} \leqslant C \| \varphi \|_{L^p(\mathbb{R}^n)}$. Then, the Young inequality requires

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$$

so that,

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$$

$$= \frac{1}{p} + \frac{n-k}{n} - 1$$

$$= \frac{1}{p} + 1 - \frac{k}{n} - 1$$

$$= \frac{1}{p} - \frac{k}{n}$$

Then,

$$\| T_\sigma \|_{L^q(\mathbb{R}^n)} \leqslant B \| f \|_{L^{r,\infty}(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}$$

$$\leqslant B \| f \|_{L^{r,\infty}(\mathbb{R}^n)} C \|$$

$$\varphi \|_{L^p(\mathbb{R}^n)}$$

$$\leqslant D \| \varphi \|_{L^p(\mathbb{R}^n)}, \quad D \geqslant 0$$

when $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.

4. Conclusion

The conclusion of this paper is the boundedness pseudo-differential operator in the Theorem 3.2. Let σ symbol in S^{-k} where $0 \leqslant k < n$, then operator $T_\sigma: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $1 < p, q < \infty$ is a bounded linear operator, with

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi) d\xi.$$

when $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$

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